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The Widom-Rowlinson model: Mesoscopic fluctuations for the critical droplet

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Abstract

The Widom-Rowlinson model is one of the rare examples of an interacting particle system in the continuum for which a vapour-liquid transition has been established rigorously. We consider the version of the model on a two-dimensional finite torus, in which the energy of a particle configuration is determined by its *halo*, i.e., the union of small discs centred at the positions of the particles. We provide a microscopic description of the *critical droplet*: the set of macroscopic states that correspond to the saddle points for the passage from a low-density vapour to a high-density liquid. In particular, we give a detailed analysis of a specific high-dimensional configuration integral associated with the critical droplet at low temperature. It turns out that the critical droplet is close to a disc of a certain deterministic radius, with a boundary that is random and consists of a large number of small discs that stick out by a small distance.

Mathematically, our analysis relies on large deviations for the volume of the halo and moderate deviations for the surface of the halo, in combination with stability properties of isoperimetric and Brunn-Minkowski inequalities. The crucial steps in the argument consist of a refined description of the statistics of the small discs forming the boundary of the critical droplet. In the present paper we are mainly concerned with mesoscopic fluctuations of the surface of macroscopic droplets, which in the physics literature are referred to as *capillary waves*. These in turn are built on microscopic fluctuations, which we analyse in [23]. We derive a sharp asymptotics for the mesoscopic fluctuations under three technical conditions, which are proved in [23]. To make the present paper self-contained, we also derive a rough asymptotics without these conditions.

Our results provide the first analysis of surface fluctuations in the Widom-Rowlinson model down to mesoscopic and microscopic precision. As such they constitute a fundamental contribution to the area of phase separation in continuum interacting particle systems from the perspective of *stochastic geometry*. At the same time they serve as a basis for [22], where we study a *dynamic* version of the Widom-Rowlinson model in which particles are randomly created and annihilated according to an infinite reservoir with a certain chemical potential. In [22] we derive the *Arrhenius law* for the vapour-liquid condensation time, in a *metastable regime* where the temperature is low and the chemical potential is high. The results in the present paper not only allow us to derive the leading order term of the condensation time – which equals the inverse temperature multiplied by the energy of the critical droplet – but also the correction order term – which equals the one-third power of the inverse temperature multiplied by a computable constant that represents the surface fluctuations of the critical droplet.

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Contents

1 Introduction, background and motivation

Section 1.1 introduces the *equilibrium* Widom-Rowlinson model. Section 1.2 states a *key target* in this model, namely, a detailed description of the fluctuations of the so-called *critical droplet*, i.e., the saddle point in the set of droplets with minimal free energy connecting the vapour state and the liquid state. In [22] we define and analyse a *dynamic* version of the Widom-Rownlinson model, for which this critical droplet appears as the *gate for the metastable transition* from the vapour state to the liquid state. Understanding the fluctuations of the critical droplet is crucial for the computation of the metastable crossover time in the dynamic model. The main goal of the present paper is a proof of the key target subject to three conditions, whose proofs are given in [23]. A weaker version of the key target is proved without the three conditions, in order to make the present paper self-contained. Section 1.3 contains an outline of the remainder of the paper.

1.1 The Widom-Rowlinson model

The Widom-Rowlinson model is an interacting particle system in $\mathbb{R}^2$ where the particles are discs with an attractive interaction. It was introduced in Widom and Rowlinson [39] to model liquid-vapour phase transitions, and is one of the rare models in the continuum for which a phase transition has been established rigorously. In the present paper we place the particles on a finite torus in $\mathbb{R}^2$.

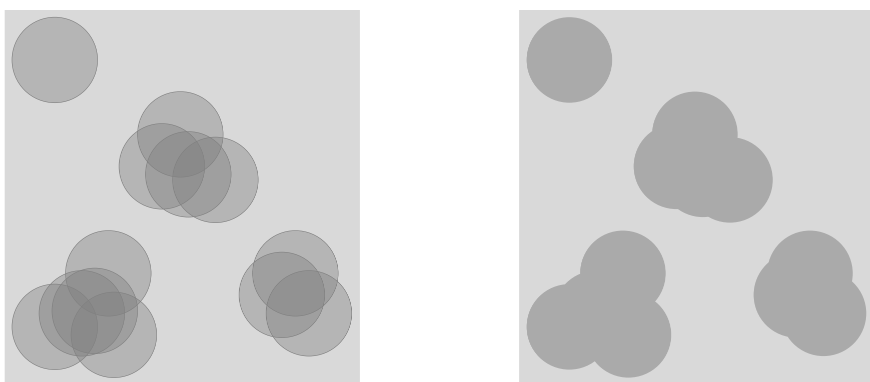

Figure 1: Picture of a particle configuration $\gamma \in \Gamma$ and its halo $h(\gamma)$.

Fix $L \in (4, \infty)$ and let $\mathbb{T} = \mathbb{T}_L = \mathbb{R}^2/(L\mathbb{Z})^2$ be the torus of side-length L. We can identify $\mathbb{T}$ with the set $[-\frac{1}{2}L, \frac{1}{2}L)^2$ after we redefine the distance by

$$\operatorname{dist}(x, y) = \inf_{k \in \mathbb{Z}} |x - y + kL|, \qquad x, y \in \mathbb{R}^2. \tag{1.1}$$

The set $\Gamma = \Gamma_{\mathbb{T}}$ of finite particle configurations in $\mathbb{T}$ is

$$\Gamma = \{\gamma \subset \mathbb{T} \colon\ N(\gamma) \in \mathbb{N}_0\}, \tag{1.2}$$

where $N(\gamma)$ denotes the cardinality of γ, i.e., particles are viewed as non-coinciding points that are indistinguishable. The *halo* of a configuration $\gamma \in \Gamma$ is defined as (see Fig. 1)

$$h(\gamma) = \bigcup_{x \in \gamma} B_2(x), \tag{1.3}$$

where $B_2(x)$ is the closed disc of radius 2 centred at $x \in \mathbb{T}$. (The reason why we choose radius 2 instead of radius 1 is explained in [22]: the one-species model arises as the projection of a two-species model.) The *energy* $E(\gamma)$ of a configuration $\gamma \in \Gamma$ is defined as

$$E(\gamma) = V(\gamma) - V_0 N(\gamma) = \left| \bigcup_{x \in \gamma} B_2(x) \right| - \sum_{x \in \gamma} |B_2(x)|, \tag{1.4}$$

where $V(\gamma) = |h(\gamma)|$ and $V_0 = |B_2(0)| = 4\pi$. The energy vanishes when the discs do not overlap, and reaches its minimal value when the discs coincide. Since $0 \geq E(\gamma) \geq -V_0[N(\gamma) - 1]$, the interaction is attractive and stable (Ruelle [30, Section 3.2]).

We define the *grand-canonical Hamiltonian* $H = H_{\mathbb{T},\lambda}$ in $\mathbb{T}$ with chemical potential λ as

$$H(\gamma) = E(\gamma) - \lambda N(\gamma), \qquad \gamma \in \Gamma. \tag{1.5}$$

The *grand-canonical Gibbs measure* $\mu_\beta = \mu_{\mathbb{T},\lambda,\beta}$ is the probability measure on Γ defined by

$$\mu_\beta(\mathrm{d}\gamma) = \frac{1}{\Xi_\beta}\, \mathrm{e}^{-\beta H(\gamma)}\, \mathbb{Q}(\mathrm{d}\gamma), \qquad \gamma \in \Gamma, \tag{1.6}$$

where $\beta \in (0,\infty)$ is the inverse temperature, $\mathbb{Q}$ is the law of the homogeneous Poisson point process on $\mathbb{T}$ with intensity 1, and $\Xi_\beta = \Xi_{\mathbb{T},\lambda,\beta}$ is the normalisation

$$\Xi_\beta = \int_\Gamma \mathbb{Q}(\mathrm{d}\gamma)\, \mathrm{e}^{-\beta H(\gamma)}. \tag{1.7}$$

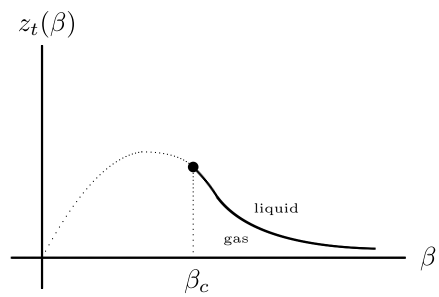

Figure 2: Picture of $\beta \mapsto z_t(\beta)$.

Write $z = \mathrm{e}^{\beta\lambda}$ to denote the chemical activity. In the thermodynamic limit, i.e., when $L \to \infty$, a *phase transition* occurs at $z = z_t(\beta)$ with (see Fig. 2)

$$z_t(\beta) = \beta\, \mathrm{e}^{-\beta V_0}, \qquad \beta > \beta_t \in (0,\infty) \tag{1.8}$$

(Ruelle [31], Chayes, Chayes and Kotecký [8]). No closed form expression is known for the critical inverse temperature β_c. We place ourselves in the *metastable regime*

$$z = \kappa z_t(\beta), \quad \kappa \in (1,\infty), \qquad \beta \to \infty \tag{1.9}$$

In other words, we choose z to lie in the liquid phase region, above the phase coexistence line in Fig. 2 representing the phase transition in the thermodynamic limit, and we let $\beta \to \infty$ and $z \downarrow 0$ in such a way that we keep close to the phase coexistence line by a fixed factor κ. For this choice the Gibbs measure in (1.6) becomes

$$\mu_\beta(\mathrm{d}\gamma) = \frac{1}{\Xi_\beta}\, (\kappa\beta)^{N(\gamma)}\, \mathrm{e}^{-\beta V(\gamma)}\, \mathbb{Q}(\mathrm{d}\gamma), \qquad \gamma \in \Gamma, \tag{1.10}$$

so that large particle numbers are favoured while large halos are disfavoured.

Let $\Pi_{\kappa\beta}$ be the homogeneous Poisson point process on $\mathbb{T}$ with intensity $\kappa\beta$. If $\mathsf{P}_{\kappa\beta}$ denotes the law of $\Pi_{\kappa\beta}$, then μ_β is absolutely continuous with respect to $\mathsf{P}_{\kappa\beta}$ with Radon-Nikodym derivative

$$\frac{\mathrm{d}\mu_\beta}{\mathrm{d}\mathsf{P}_{\kappa\beta}}(\gamma) = \frac{\exp(-\beta V(\gamma))}{\int_\Gamma \exp(-\beta V)\, \mathrm{d}\mathsf{P}_{\kappa\beta}}, \qquad \gamma \in \Gamma. \tag{1.11}$$

1.2 A key target: the critical droplet

For $\kappa \in (1,\infty)$, abbreviate (see Fig. 3),

$$\Phi_\kappa(R) = \pi R^2 - \kappa\pi(R-2)^2, \quad R \in [2,\infty), \qquad R_c(\kappa) = \frac{2\kappa}{\kappa-1}. \tag{1.12}$$

Throughout the paper, $\kappa \in (1,\infty)$ and $\frac{1}{2}L > R_c(\kappa)$ are fixed (recall that L is the linear size of the torus $\mathbb{T} = \mathbb{T}_L$). Define

$$\Phi(\kappa) = \Phi_\kappa(R_c(\kappa)) = \frac{4\pi\kappa}{\kappa-1}, \qquad \Psi(\kappa) = \frac{s\kappa^{2/3}}{\kappa-1}, \qquad s \in \mathbb{R}, \tag{1.13}$$

where s is a constant that will be identified below and that does not depend on κ.

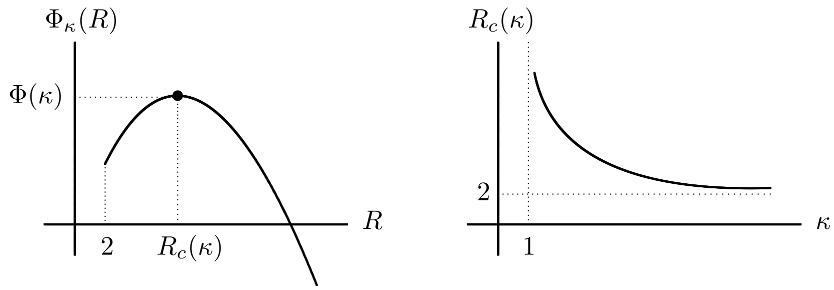

Figure 3: Picture of $R \mapsto \Phi_\kappa(R)$ for fixed $\kappa \in (1,\infty)$ and $\kappa \mapsto R_c(\kappa)$.

Fix $C \in (0,\infty)$, abbreviate $\delta(\beta) = \beta^{-2/3}$, and define

$$\begin{aligned} I(\kappa,\beta;C) &= \int_\Gamma \mathbb{Q}(\mathrm{d}\gamma)\,(\kappa\beta)^{N(\gamma)}\,\mathrm{e}^{-\beta V(\gamma)}\,\mathbf{1}_{\{|V(\gamma)-\pi R_c(\kappa)^2|\leq C\delta(\beta)\}} \\ &= \Xi_\beta\,\mu_\beta\Big(|V(\gamma)-\pi R_c(\kappa)^2| \leq C\delta(\beta)\Big). \end{aligned} \tag{1.14}$$

As shown in [22], $I(\kappa,\beta;C)$ appears as the leading order term in a computation of the Dirichlet form associated with a *dynamic* version of the Widom-Rowlinson model. In this dynamic version, the special role of the critical disc $B_{R_c(\kappa)}(0)$ becomes apparent through the fact that the set $\{\gamma \in \Gamma\colon\, |V(\gamma)-\pi R_c(\kappa)^2| \leq C\delta(\beta)\}$ forms the *gate* for the metastable transition from the gas phase ('$\mathbb{T}$ empty') to the liquid phase ('$\mathbb{T}$ full') in the metastable regime (1.9). The main ingredient in [22] is the following sharp asymptotics.

TARGET: For C large enough and $\beta \to \infty$,

$$I(\kappa,\beta;C) = \mathrm{e}^{-\beta\,\Phi(\kappa)+\beta^{1/3}\Psi(\kappa)+o(\beta^{1/3})}. \tag{1.15}$$

The above target is a statement about a *restricted equilibrium*: it provides a sharp estimates for the probability that the halo has a volume that lies inside an interval of width $C\beta^{-2/3}$ around the volume of the critical disc $B_{R_c(\kappa)}(0)$. By writing

$$\Phi(\kappa) = -(\kappa-1)\pi(R_c(\kappa)-2)^2 + \big[\pi R_c(\kappa)^2 - \pi(R_c(\kappa)-2)^2\big], \tag{1.16}$$

we see that we may think of $\Phi(\kappa)$ as (a leading order approximation of) the *free energy of the critical droplet*, consisting of the bulk free energy and the surface tension, and of $\Psi(\kappa)$ as (a leading order approximation of) the *entropy* associated with the fluctuations of the surface of the critical droplet, which plays the role of a correction term to the free energy. We will see that there are order β discs inside the critical droplet and order $\beta^{1/3}$ discs touching its boundary (see Fig. 4). We remark that β is

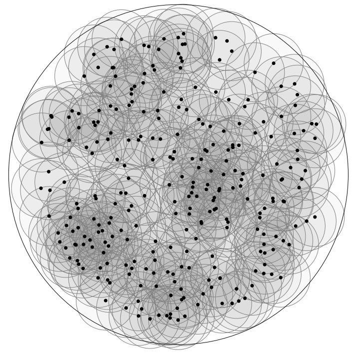

Figure 4: In [22] we find that the critical droplet in the metastable regime (1.9) is close to a disc of radius $R_c(\kappa)$ and has a random boundary that fluctuates within a narrow annulus whose width shrinks to zero as $\beta \to \infty$. Order β discs lie inside, order $\beta^{1/3}$ discs touch the boundary. In the physics literature, mesoscopic fluctuations of the surface of macroscopic droplets in the continuum are referred to as *capillary waves* (see Stillinger and Weeks [38]).

to be viewed as a dimensionless quantity, i.e., the inverse temperature divided by some unit of energy. Otherwise, its fractional powers would not make sense.

The goal of the present paper is to provide a proof of (1.15), *subject to three conditions* that are settled in [23]. To make the present paper self-contained, we also prove a rough asymptotics that does not require these conditions, but still provides the correct order of magnitude for the entropy term, with bounds on the constant s in (1.13). Along the way we will see that

$$\Xi_\beta = \mathrm{e}^{-\beta(1-\kappa)|\mathbb{T}|+o(1)}. \tag{1.17}$$

Since $\Psi(\kappa)$ does not depend on C, we may view (1.15) as a weak large deviation principle for the random variable $\beta^{2/3}|V - \pi R_c^2(\kappa)|$ under the Gibbs measure in (1.10). The rate is $\beta^{1/3}$ and the rate function is degenerate, being equal to the constant $\Psi(\kappa)$. This degeneracy reflects the fact that the radius of the critical droplet is close to $R_c(\kappa)$, for which $\Phi'_\kappa(R_c(\kappa)) = 0$.

1.3 Outline

In Section 2 we present our main theorems: a large deviation principle for the halo shape and the halo volume, and weak moderate deviations for the halo volume close to the critical droplet. These theorems are the main input for the analysis of the dynamic Widom-Rowlinson model in [22]. Section 3 proves the two large deviation principles, as well as certain isoperimetric inequalities that play a crucial role throughout the paper. In Section 4 we provide the heuristics behind the proof of the main theorems, which is carried out in Sections 5–8. Section 5 focusses on approximations of certain key geometric functionals, which are crucial for the analysis of the moderate deviations. Section 6 represents moderate deviation probabilities in terms of geometric surface integrals and introduces auxiliary random processes that are needed for the description of the fluctuations of the surface of the critical droplet. Section 7 contains various preparations involving exponential functionals of the auxiliary random variables. Section 8 uses these preparations, in combination with the geometric properties derived in Sections 5–7, to prove the moderate deviations for the halo volume close to the critical droplet.

The results in Sections 3–8 lead to a description of the *mesoscopic* fluctuations of the surface of the critical droplet in terms of a certain constrained Brownian bridge and quantifies the cost of moderate deviations for the surface free energy of droplets. The proof relies on three conditions involving the *microscopic* fluctuations of the surface of the critical droplet, whose proofs are given [23]. To make the present paper self-contained, we also prove a rough moderate deviation estimate that does not need the three conditions.

2 Main theorems

This section formulates and discusses our main theorems. In Section 2.1 we state large deviation principles for the halo shape (Theorem 2.1) and the halo volume (Theorem 2.3), and show that the corresponding rate functions are linked via an isoperimetric inequality (Theorem 2.2). In Section 2.2 we formulate a conjecture (Conjecture 2.4) about moderate deviations for the halo volume, and state a sharp asymptotics that settles a version of this conjecture for volumes that are close to the volume of the critical droplet (Theorem 2.5). This sharp asymptotics settles the target formulated in Section 1.2, subject to three conditions (Conditions (C1)–(C3) below), whose proof is given in [23]. In order to make our paper self-contained, we also prove a rougher asymptotics (Theorem 2.6), which does not require the conditions and still provides the correct order of the correction term, with explicit bounds on the constant. In Section 2.3 we place our results in a broader context and explain why they open up a new window in the area of *stochastic geometry for interacting particle systems*.

For background on large deviation theory, see e.g. Dembo and Zeitouni [11] or den Hollander [21].

2.1 Large deviation principles and isoperimetric inequalities

Admissible sets. Let $\mathcal{F}_{\mathbb{T}}$ be the family of non-empty closed (and hence compact) subsets of the torus $\mathbb{T}$. We equip $\mathcal{F}_{\mathbb{T}}$ with the Hausdorff metric

$$\begin{aligned} d_{\mathrm{H}}(F_1,F_2) &= \max\Big\{\max_{x\in F_1}\mathrm{dist}(x,F_2),\max_{x\in F_2}\mathrm{dist}(x,F_1)\Big\} \\ &= \min\Big\{\varepsilon\geq 0\colon\ F_1\subset F_2+\varepsilon B_1(0), F_2\subset F_1+\varepsilon B_1(0)\Big\}, \qquad F_1,F_2\neq\emptyset, \end{aligned} \tag{2.1}$$

where $\mathrm{dist}(x,F)=\min_{y\in F}\mathrm{dist}(x,y)$. This turns $\mathcal{F}_{\mathbb{T}}$ into a compact metric space (Matheron [28, Propositions 12.2.1, 1.4.1, 1.4.4], Schneider and Weil [32, Theorems 12.2.1, 12.3.3]). Let $\mathcal{S}\subset\mathcal{F}_{\mathbb{T}}$ be the collection of all sets that are ($\mathbb{T}$-)*admissible*, i.e.,

$$\mathcal{S}_{\mathbb{T}}=\{S\subset\mathbb{T}\colon\ \exists\, F \text{ such that } h(F)=S\}, \tag{2.2}$$

where $h(F)=\cup_{x\in F}B_2(x)$ is the halo of F. In Section 3.1 we will see that there is a unique maximal F such that $h(F)=S$, which we denote by S^- and which equals $S^-=\{x\in S\colon\ B_2(x)\subset S\}$.

Obviously, not every closed set is admissible. For example, when we form 2-halos we round off corners, and so a shape with sharp corners cannot be in $\mathcal{S}$. Also note that $S^-\neq\emptyset$ whenever S is admissible: S necessarily contains at least one disc $B_2(x)$ with $x\in S$. In the following, we typically omit the subscript referring to the torus $\mathbb{T}$.

Large deviation principles. Define

$$J(S)=|S|-\kappa|S^-|,\qquad S\in\mathcal{S}, \tag{2.3}$$

and

$$I(S)=J(S)-\inf_{\mathcal{S}}J. \tag{2.4}$$

We view the halo $h(\gamma)$ as a random variable with values in the space $\mathcal{S}$, topologized with the Hausdorff distance. Note that $\inf_{\mathcal{S}}J=(1-\kappa)|\mathbb{T}|$ because $\kappa\in(1,\infty)$.

Theorem 2.1 (Large deviation principle for the halo shape).
The family of probability measures $(\mu_\beta(h(\gamma)\in\cdot))_{\beta\geq 1}$ satisfies the LDP on $\mathcal{S}$ with speed β and with good rate function I.

Informally, Theorem 2.1 says that

$$\mu_\beta\big(h(\gamma)\approx S\big)\approx\exp\big(-\beta I(S)\big),\qquad \beta\to\infty. \tag{2.5}$$

The contraction principle suggests a large deviation principle for the halo volume. To formulate this, we first state a minimisation problem. The condition $R\in(2,\frac{1}{2}L)$ below ensures that that the effect of the periodic boundary conditions on the torus $\mathbb{T}$ is not felt.

Theorem 2.2 (Minimisers of rate function for halo volume)**.**

(1) *For every* $R \in (2, \frac{1}{2}L)$*,*

$$\min\left\{|S| - \kappa|S^-|\colon\ S \in \mathcal{S}, |S| = \pi R^2\right\} = \pi R^2 - \kappa\pi(R-2)^2 \tag{2.6}$$

and the minimisers are the discs of radius R*.*

(2) *The minimisers are stable in the following sense: There exists an* $\varepsilon_0 > 0$ *such that if* $0 < \varepsilon < \varepsilon_0$ *and* $S \in \mathcal{S}$ *satisfies*

$$\big(|S| - \kappa|S^-|\big) - \big(\pi R^2 - \kappa\pi(R-2)^2\big) \leq \pi\kappa\varepsilon \quad \textit{with} \quad |S| = \pi R^2, \tag{2.7}$$

then S^- *is connected with connected complement (simply connected as a subset of* $\mathbb{R}^2$*), and*

$$d_{\mathrm{H}}(\partial S, \partial B_R) \leq \tfrac{3}{2}\sqrt{R\varepsilon}, \tag{2.8}$$

where d_{H} *denotes the Hausdorff distance.*

Theorem 2.2 is a powerful tool because it shows that *the near-minimers of the halo rate function are close to a disc and have no holes inside.* In particular, it tells us that

$$I(B_R) = \Phi_\kappa(R) - (1-\kappa)|\mathbb{T}|, \tag{2.9}$$

and allows us to describe the large deviations of the halo volume.

Theorem 2.3 (Large deviation principle for the halo volume)**.**
The family of probability measures $(\mu_\beta(V(\gamma) \in \cdot\,))_{\beta\geq 1}$ *satisfies the LDP on* $[0,\infty)$ *with speed* β *and with good rate function* I^* *given by*

$$I^*(A) = \inf\{I(S)\colon\ |S| = A\}, \qquad A \in [0,\infty). \tag{2.10}$$

Informally, Theorem 2.3 says that

$$\mu_\beta\big(V(\gamma) \approx A\big) \approx \exp\big(-\beta I^*(A)\big). \tag{2.11}$$

For every $R \in (2, \frac{1}{2}L)$, we have

$$I^*(\pi R^2) = I(B_R). \tag{2.12}$$

2.2 Near the critical droplet: moderate deviations

Fluctuations of the halo volume. The function $R \mapsto I(B_R)$ is maximal at

$$R_c = \frac{2\kappa}{\kappa - 1}. \tag{2.13}$$

We assume that $L > \frac{1}{2}R_c$. In the *dynamic* Widom-Rowlinson model that we introduce and analyse in [22], R_c plays the role of the radius of the *critical droplet* for the metastable crossover from an empty torus to a full torus in the limit as $\beta \to \infty$. We therefore zoom in on a neighborhood of the critical droplet. The large deviation principle yields the statement

$$\mu_\beta\Big(|V(\gamma) - \pi R_c^2| \leq \varepsilon\Big) = \exp\Big(-\beta \min_{\substack{A\in[0,\infty):\\ |A-\pi R_c^2|\leq\varepsilon}} I^*(A) + o(\beta)\Big), \qquad \beta \to \infty, \tag{2.14}$$

for $\varepsilon > 0$ fixed. We would like to take $\varepsilon = \varepsilon(\beta) \downarrow 0$, for which we need a stronger property.

Conjecture 2.4 (Weak moderate deviations for the halo volume)**.**
There exists a function $\Psi_R\colon \mathbb{R} \to \mathbb{R}$ *such that*

$$\begin{aligned}
\limsup_{\beta\to\infty} \tfrac{1}{\beta^{1/3}} \log\Big\{\mathrm{e}^{\beta I(B_R)}\mu_\beta\Big(\beta^{2/3}\big[V(\gamma) - \pi R^2\big] \in K\Big)\Big\} \quad &\leq \quad \sup_K \Psi_R \quad \forall\, K \subset \mathbb{R} \textit{ compact},\\
\liminf_{\beta\to\infty} \tfrac{1}{\beta^{1/3}} \log\Big\{\mathrm{e}^{\beta I(B_R)}\mu_\beta\Big(\beta^{2/3}\big[V(\gamma) - \pi R^2\big] \in O\Big)\Big\} \quad &\geq \quad \sup_O \Psi_R \quad \forall\, O \subset \mathbb{R} \textit{ open}.
\end{aligned} \tag{2.15}$$

Conjecture 2.4 has the flavor of a *weak large deviation principle* on scale $\beta^{-2/3}$ with speed $\beta^{1/3}$. For $R = R_c$ we expect the function Ψ_{R_c} to be constant. In Theorem 2.5 below we establish a version of this claim, with $\Psi_{R_c} \equiv \Psi(\kappa)$, the entropy defined in (1.13). In what follows we first state a theorem (Theorem 2.5 below) that relies on *three conditions* involving an *effective interface model* whose proof is given in [23]. Afterwards we state a weaker version (Theorem 2.6 below) that provides upper and lower bounds, whose proof is fully completed in the present paper. To state these theorems we need some additional notation.

Notation. Let $S = h(\gamma)$ be the halo of some configuration γ. The boundary of S consists of a union of circular arcs that are disjoint except for their endpoints. We call the centres $z_1, \dots, z_n$ of these circles the *boundary points* of S and we say that $z = (z_1, \dots, z_n)$ is a *connected outer contour* if there exists a halo S with a simply connected 2-interior S^- having exactly these boundary points (see Fig. 5). Let

$$\mathcal{O} = \text{ set of connected outer contours.} \tag{2.16}$$

Later on we parametrize the boundary points $z_1, \dots, z_n$ of an approximately disc-shaped droplet in polar coordinates. We will see that, roughly, we may think of the angular coordinates $t_1, \dots, t_n$ as the points of an angular point process, and of the radii $r_1, \dots, r_n$ as the values of a Gaussian process evaluated at those random angles. To make this picture more precise, we need to introduce auxiliary processes.

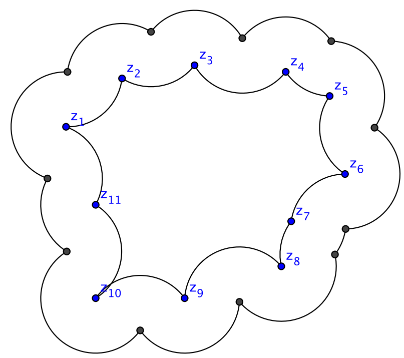

Figure 5: Circular arcs of the *outer boundary* $\partial h(\gamma)$ and the *inner boundary* $\partial h(\gamma)^-$ with $z(\gamma)$ consisting of 11 boundary points. Here, $h(\gamma)^-$ denotes the 2-interior of $h(\gamma)$.

Let $(W_t)_{t\geq 0}$ be standard Brownian motion starting in 0, and let

$$(\widetilde{W}_t)_{t\in[0,2\pi]}, \qquad \widetilde{W}_t = W_t - \frac{t}{2\pi} W_{2\pi}, \tag{2.17}$$

be standard Brownian bridge on $[0, 2\pi]$. Consider the process

$$(B_t)_{t\in[0,2\pi]}, \qquad B_t = \widetilde{W}_t - \frac{1}{2\pi}\int_0^{2\pi} \widetilde{W}_s \mathrm{d}s, \tag{2.18}$$

called the *mean-centred Brownian bridge* (Deheuvels [10]). Set

$$\lambda(\beta) = G_\kappa \beta^{1/3}, \quad G_\kappa = \frac{(2\kappa)^{2/3}}{\kappa - 1}. \tag{2.19}$$

Let

$$\begin{aligned} &\mathcal{T} = \text{Poisson point process on } [0, 2\pi) \text{ with intensity } \lambda(\beta), \\ &N = |\mathcal{T}| = \text{cardinality of } \mathcal{T}. \end{aligned} \tag{2.20}$$

Thus, N is a Poisson random variable with parameter $2\pi\lambda(\beta) = 2\pi G_\kappa \beta^{1/3}$. We assume that $(B_t)_{t\in[0,2\pi]}$ and $\mathcal{T}$ are defined on a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and they are independent. Conditional on the event $\{N = n\}$, we may write $\mathcal{T} = \{T_i\}_{i=1}^n$ with $0 \le T_1 < \cdots < T_n < 2\pi$, and define

$$\Theta_i = T_{i+1} - T_i,\ 1 \le i \le n, \qquad \Theta_n = (T_1 + 2\pi) - T_n. \tag{2.21}$$

Note that $\Theta_i \ge 0$, $1 \le i \le n$, and $\sum_{i=1}^n \Theta_i = 2\pi$. For $m \in \mathbb{R}$, set

$$Z^{(m)} = \{Z_i^{(m)}\}_{i=1}^N \tag{2.22}$$

with

$$Z_i^{(m)} = \Big(r_i^{(m)} \cos T_i, r_i^{(m)} \sin T_i\Big), \quad r_i^{(m)} = (R_c - 2) + \frac{m + B_{T_i}}{\sqrt{(\kappa-1)\beta}}, \qquad 1 \le i \le N. \tag{2.23}$$

Later we will see that the natural reference measure for the angles t_i is not the Poisson process $\mathcal{T}$ but a periodic version of a renewal process, which is conveniently constructed by tilting the distribution of the Poisson process $\mathcal{T}$. The precise expression for the exponential tilt will be derived later. Set

$$\widehat{Y}_0 = \frac{1}{2} \sum_{i=1}^N \log\big(2\pi\beta^{1/3} G_\kappa \Theta_i\big), \quad \widehat{Y}_1 = \frac{1}{24} \sum_{i=1}^N \big(\beta^{1/3} G_\kappa \Theta_i\big)^3, \tag{2.24}$$

and consider the tilted probability measure $\widehat{\mathbb{P}}$ on $(\Omega, \mathscr{F}, \mathbb{P})$ defined by

$$\widehat{\mathbb{P}}(A) = \frac{\mathbb{E}[\exp(\widehat{Y}_0 - \widehat{Y}_1)\mathbf{1}_A]}{\mathbb{E}[\exp(\widehat{Y}_0 - \widehat{Y}_1)]}, \qquad A \subset \Omega \text{ measurable.} \tag{2.25}$$

Finally let $\tau_* \in \mathbb{R}$ be the unique solution to the equation

$$\int_0^\infty \sqrt{2\pi u} \exp\Big(-\tau_* u - \frac{u^3}{24}\Big)\, \mathrm{d}u = 1. \tag{2.26}$$

The change of variables $s = u^3/24$ together with $\int_0^\infty s^{-1/2} \mathrm{e}^{-s} \mathrm{d}s = \Gamma(\frac{1}{2}) = \sqrt{\pi}$ yields

$$\int_0^\infty \sqrt{2\pi u} \exp\left(-\frac{u^3}{24}\right)\, \mathrm{d}u = \frac{4\pi}{\sqrt{3}} > 1, \tag{2.27}$$

and so $\tau_* > 0$. In the sequel we use the notation $\overline{u_i} = \frac{1}{2}(u_i + u_{i+1})$, $1 \le i \le N$.

Conditions. Our main theorem builds on three conditions whose validity is proven in [23]:

(C1) The limit

$$-\, c_{**} = \lim_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log \widehat{\mathbb{P}}(Z^{(0)} \in \mathcal{O}) \tag{2.28}$$

exists and is of the form

$$c_{**} = 2\pi G_\kappa \tau_{**} \tag{2.29}$$

for some $\tau_{**} > 0$ that does not depend on κ.

(C2) The change from $Z^{(0)}$ to $Z^{(m)}$ does not affect (C1) when m is not too large:

$$\lim_{\beta\to\infty} \frac{1}{\beta^{1/3}} \sup_{|m| = O(\beta^{1/6})} \left| \log \frac{\widehat{\mathbb{P}}(Z^{(m)} \in \mathcal{O})}{\widehat{\mathbb{P}}(Z^{(0)} \in \mathcal{O})} \right| = 0. \tag{2.30}$$

(C3) Let

$$D_1 = \frac{1}{\sqrt{\pi}} \sum_{i=1}^{N} \Theta_i \overline{B_{T_i} \cos T_i}, \quad D_1^* = \frac{1}{\sqrt{\pi}} \sum_{i=1}^{N} \Theta_i \overline{B_{T_i} \sin T_i} \tag{2.31}$$

and

$$\chi = \sum_{i=1}^{N} \Theta_i \overline{B_{T_i}}^2 - D_1^2 - D_1^{*2}. \tag{2.32}$$

Then, for $\delta > 0$ sufficiently small,

$$\limsup_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log \frac{\widehat{\mathbb{E}}[\exp(\frac{1}{2}(1+\delta)\chi)\mathbf{1}_{\{Z^{(m)}\in\mathcal{O}\}}]}{\widehat{\mathbb{P}}(Z^{(m)}\in\mathcal{O})} = 0, \tag{2.33}$$

uniformly in $|m| = O(\beta^{1/6})$.

Condition (C1) comes from the fact that for each of the $N \asymp \beta^{1/3}$ boundary points there is a constraint in terms of the two neighbouring boundary points that must be satisfied in order for the corresponding 2-disc to touch the boundary of the critical droplet. The constant τ_{**} is related to the free energy of an *effective interface model*. Condition (C2) says that the constraint imposed by condition (C1) is not affected by *small dilations* of the critical droplet, and implies that the free energy of the effective interface model is Lipschitz under small perturbations. Condition (C3) says that the first Fourier coefficient of the surface of the critical droplet is small. The term χ represents an energetic and entropic reward for the droplet boundary to fluctuate away from ∂B_{R_c}. We require that this reward – which may be thought of as a background potential in the effective interface model – does not affect the microscopic free energy of the droplet.

Main theorem: sharp asymptotics. We are now ready to formulate our main theorem.

Theorem 2.5 (Moderate deviations)**.** *Suppose that conditions* (C1)–(C3) *hold. Then, for C large enough and $\beta \to \infty$,*

$$\mu_\beta\Big(|V(\gamma) - \pi R_c^2| \le C\beta^{-2/3}\Big) = e^{-\beta I(B_{R_c}) + \beta^{1/3}\Psi(\kappa) + o(\beta^{1/3})}, \tag{2.34}$$

where

$$I(B_{R_c}) = \Phi(\kappa) - (1-\kappa)|\mathbb{T}|, \qquad \Psi(\kappa) = 2\pi G_\kappa(\tau_* - \tau_{**}). \tag{2.35}$$

In view of (1.14) and (1.17), Theorem 2.5 settles the target in (1.15) subject to conditions (C1)–(C3).

Rough asymptotics. Without conditions (C1)–(C3) we can still prove the following rougher asymptotics, which makes our paper self-contained.

Theorem 2.6 (Moderate deviation bounds)**.** *For C large enough,*

$$\begin{aligned} \limsup_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log\Big\{\mathrm{e}^{\beta I(B_{R_c})}\mu_\beta\Big(|V(\gamma) - \pi R_c^2| \le C\beta^{-2/3}\Big)\Big\} &\le 2\pi G_\kappa \tau_*, \\ \liminf_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log\Big\{\mathrm{e}^{\beta I(B_{R_c})}\mu_\beta\Big(|V(\gamma) - \pi R_c^2| \le C\beta^{-2/3}\Big)\Big\} &\ge 2\pi G_\kappa (\tau_* - c), \end{aligned} \tag{2.36}$$

with $c \in [0,\infty)$ some constant (that may depend on κ).

Theorem 2.5 provides a full description of the fluctuations of the surface of the critical droplet in the Widom-Rowlinson model in the metastable regime (1.9). It makes fully precise and rigorous the heuristic arguments for capillary waves that are put forward in Stillinger and Weeks [38]. In the proof of Theorem 2.5 in Sections 5–8 we will see that the boundary points are given by (2.23), with $(B_t)_{t\in[0,2\pi]}$ the mean-centred Brownian bridge, $\mathcal{T}$ the Poisson point process on $[0,2\pi]$ with intensity $2\pi G_\kappa \beta^{1/3}$ tilted via the probability measure in (2.25), and $|m| = O(\beta^{-1/6})$ (which is negligible). We will also see that the effect of centring of the droplet is that the discrete Fourier coefficients defined in (2.31) are asymptotically vanishing.

2.3 Context

In this section we place our main theorems in the broader context of stochastic geometry and continuum statistical physics. For general overviews we refer the reader to Chiu, Stoyan, Kendall and Mecke [9], respectively, Georgii, Häggström and Maes [18].

Large and moderate deviations for confined point processes. There is a rich literature on limit laws for high-intensity point processes and extremal points, and also on the high-intensity Widom-Rowlinson model. Let us explain why Theorem 2.5 is considerably more involved than the theorems encountered in that literature. We first review two papers in the context of stochastic geometry that are close in spirit to our results and discuss the differences. The results from the point process literature are valid in higher dimensions $d \geq 2$ as well, but for simplicity we present the summary for $d = 2$ only.

Consider a disc of fixed radius $R > 2$ and a homogeneous Poisson point process with intensity α in $B_{R-2}(0)$. The set $B_R(0) \setminus h(\Pi_\alpha)$ is called the *vacant set*, the volume $|B_R(0) \setminus h(\Pi_\alpha)|$ is called the *defect volume*. A point $x \in \Pi_\alpha$ is *extremal* when $h(\Pi_\alpha \setminus x) \subsetneq h(\Pi_\alpha)$. Write $\xi(x, \Pi_\alpha)$ for the indicator that x is extremal in Π_α. (The problem has been also studied for general compact sets K instead of $B_2(0)$, in which case the points are called *K-maximal*.) The following results are available in the limit as $\alpha \to \infty$:

(I) Schreiber [34] focusses on a Boolean model that is a variation on the Widom-Rowlinson model, in which to each point of the Poisson point process a random closed set called *grain* is attached. Grains are deterministic compact convex smooth sets satisfying certain conditions, namely, they are contained in $B_R(0)$, are twice differentiable on the boundary, and are parametrized by the point closest to the boundary of $B_R(0)$. The following results are proved in [34]:

(i) A law of large numbers for the defect volume:

$$\lim_{\alpha\to\infty} \alpha^{2/3}\mathbb{E}\Big[|B_R(0) \setminus h(\Pi_\alpha)|\Big] = c_\infty \quad \text{a.s.} \tag{2.37}$$

(ii) A moderate deviation estimate: if E_α is the expected defect volume, then for all $\eta > 0$ and some $\widehat{I}(\eta) > 0$,

$$\mathbb{P}\Big(|B_R(0) \setminus h(\Pi_\alpha)| > (1+\eta)E_\alpha\Big) \leq \exp\Big(-\alpha^{1/3}\widehat{I}(\eta) + o(\alpha^{1/3})\Big). \tag{2.38}$$

The limit $\lim_{\eta\to\infty} \frac{1}{\eta}\widehat{I}(\eta) \in \mathbb{R}$ exists. An LDP is proved for a point process whose law is a Gibbsian modification of a Poisson point process with a hard-core Hamiltonian, which describes the one-color process in the two-color Widom-Rowlinson model. This result is close to our LDP, the main difference being that the minimization is much easier than in our model.

(II) Schreiber [35] considers a homogeneous Poisson point process with intensity α restricted to $[0,1]^d$ (which by abuse of notation we again call Π_α) and proves the following: $h_r(\Pi_\alpha) = \cup_{x\in\Pi_\alpha} B_r(x)$ satisfies the full LDP on $L^1([0,1]^d$ with rate $r\alpha$ and with good rate function $\mathcal{P}$, the so-called Caccioppoli perimeter. However, this is proved in a specific limit for α and $r = r(\alpha)$ jointly, namely,

$$\lim_{\alpha\to\infty} r(\alpha) = 0, \quad \lim_{\alpha\to\infty} \frac{\alpha r(\alpha)^d}{\log(1/r(\alpha))} = \infty, \tag{2.39}$$

(i.e., the large-volume limit and the high-intensity limit are taken simultaneously), while in our setting the volume of the system and the radius of the discs are kept fixed. Furthermore, the parameters of the model are taken to be *on* the coexistence line. In our case this would amount to taking the limit $\kappa \downarrow 1$ instead of fixing $\kappa \in (1,\infty)$ arbitrarily.

In Section 3 we give a full description of large deviations for *arbitrary* droplets, and in Sections 5–8 of moderate deviations for *near-critical* droplets. The fluctuations that we consider are *two-sided*, i.e., the droplets are not confined to an ambient disk $B_R(0)$ but rather live on a torus. [FdH: OKAY?]

Interface literature. Another popular concept in stochastic geometry is that of *random convex polytope* and its relation with the *paraboloid growth process* (see Schreiber and Yukich [36], Calka, Schreiber and Yukich [7]). The former is the convex hull of $K \cap \Pi_\alpha$ (with K a smooth convex set in $\mathbb{R}^d$), the latter is a growth model for interfaces that is studied because it provides information on the asymptotic behaviour as $\alpha \to \infty$ of geometric functionals of random polytopes, in particular, on the distribution of K-*maximal* points. While we marginally touch on these concepts in the present paper, they play an important role in [23], where we discuss the microscopic fluctuations of the surface of the critical droplet. There we prove that, upon rescaling of the random variables describing the boundary of the critical droplet, the effective microscopic model is given by a modification of the *paraboloid hull process*, which is connected to the *paraboloid growth process*. In [23], the conditions (C1)–(C3) in Theorem 2.5 are formulated in the context of paraboloid hull processes, and are proven there.

The literature on stochastic interfaces is large, especially for phase boundaries separating two phases. In statistical mechanics interface analyses have been successfully carried out for discrete systems, such as the Ising model. Higuchi, Murai and Wang [20] adapted to the two-dimensional Widom-Rowlinson model what was done in Higuchi [19] for the two-dimensional Ising model. As for the Ising model, the interface is well approximated by that of the so-called Solid-on-Solid model. The results in [20] concerns limiting properties of the continuous random processes that model the fluctuations of the interface in the direction orthogonal to the line connecting the two end points of the phase boundary. Diffusive scaling of the interface is shown, with Brownian bridge appearing as the limit. However, the results in [20] are again only for parameters *on* the coexistence line, which in our case amounts to taking the limit $\kappa \downarrow 1$.

Further variations on the Widom-Rowlinson model. Several further variations on the Widom-Rowlinson model have been considered in the literature. One is the so-called area-interaction point process considered in Baddeley and van Lieshout [4], where the probability density of a point configuration depends on the area of $h(\Pi_\alpha)$ through a parameter γ. When $\gamma = 1$, the Widom-Rowlinson model is recovered. Another is the so-called quermass-interaction processes introduced in Kendall, van Lieshout and Baddeley [26], where a Boolean model interacting via a linear combination of Minkowski functionals generalizes the above area-interaction. In both these papers the problem of well-posedness of the processes are addressed via a proof of integrability and stability in the sense of Ruelle [30], but no result in terms of large deviations are obtained.

Another generalization regards the multi-color Widom-Rowlinson model (see Chayes, Chayes and Kotecký [8]): q colors are considered and random radii r_i, $1 \le i \le q$, are attached to any color. In Dereudre and Houdebert [12] the case with non-integrable random radii is studied, and a different type of phase transition is proved.

3 Proof of large deviation principles and isoperimetric inequalities

In this section we prove Theorems 2.1–2.3. Section 3.1 takes a closer look at the properties of admissible sets (Lemmas 3.1–3.2). Section 3.2 gives the proof of Theorem 2.2. The proof requires two isoperimetric inequalities (Lemmas 3.3–3.4), which are analogues of the classical isoperimetric and Bonnesen inequalities, and imply that the minimisers of I in (2.4) are discs and that the difference of I with its minimum can be quantified in terms of the Hausdorff distance to these discs. Section 3.3 proves the large deviation principle for the centres of the 2-discs in the Widom-Rowlinson model (Proposition 3.5), and uses this to prove Theorems 2.1 and 2.3.

3.1 Properties of admissible sets

Write

$$F^+ = F + B_2(0) = \bigcup_{x \in B_2(0)} (F + x) = h(F) \tag{3.1}$$

for the 2-halo of $F \in \mathcal{F}_{\mathbb{T}}$ (Minkowski addition) and

$$F^- = F \ominus B_2(0) = \bigcap_{x \in B_2(0)} (F + x) = \{x \in F\colon\ B_2(x) \subset F\}, \tag{3.2}$$

for the 2-interior of $F \in \mathcal{F}_{\mathbb{T}}$ (Minkowski subtraction). In integral geometry, the sets F^+ and F^- are called the *dilation* and the *erosion* of F, respectively. Note that the erosion and subsequent dilation of a set F is contained in F, i.e.,

$$(F^-)^+ = \bigcup_{B_2(x) \subset F} B_2(x) \subset F \tag{3.3}$$

(Matheron [28, Section 1.5]). See Lemma 3.1(1) below.

Note that $(F^-)^+$ is not necessarily equal to F. We use $\mathcal{S} \subset \mathcal{F}_{\mathbb{T}}$ to denote the collection of all sets for which equality holds and call them ($\mathbb{T}$-)*admissible* (*open with respect to* $B_2(0)$ in the terminology of integral geometry), i.e.,

$$\mathcal{S}_{\mathbb{T}} = \{S \subset \mathbb{T}\colon\ (S^-)^+ = S\} = \{S \subset \mathbb{T}\colon\ S = (S \ominus B_2(0)) + B_2(0)\}. \tag{3.4}$$

In the following, we typically omit the subscript referring to the torus $\mathbb{T}$, by writing $\mathcal{F}_{\mathbb{T}} = \mathcal{F}$, $\mathcal{S}_{\mathbb{T}} = \mathcal{S}$. There is another useful characterisation of admissible sets: $S \in \mathcal{S}$ if and only if it is the 2-halo of some $F \in \mathcal{F}$, $S = F^+$ (see Lemma 3.1(2) below). Obviously, not every closed set is admissible. For example, when we form 2-halos we round off corners, and so a shape with sharp corners cannot be in $\mathcal{S}$. Also note that $S^- \neq \emptyset$ whenever S is admissible: S necessarily contains at least one disc $B_2(x)$ with $x \in S$.

In this section we summarise some known properties of admissible sets in a setting that will be needed later. The proofs of these properties rely on various sources. Below we only quote appropriate references, and when instructive we supply a short proof.

A key property is that for any set $S \in \mathcal{S}$ such that S^- is connected and S is simply connected, the set S^- is of reach at least 2. Recall that the reach of a set $F \in \mathcal{F}$ is

$$\mathrm{reach}(F) = \sup\big\{r \geq 0\colon\ \text{for any } x \in F + B_r(0) \text{ there exist a unique } y \in F \text{ nearest to } x\big\}. \tag{3.5}$$

Lemma 3.1. (1) *If $F \in \mathcal{F}$, then $(F^-)^+ \subset F$.*

(2) *$S \in \mathcal{S}$ if and only if S is the 2-halo of some $F \in \mathcal{F}$, i.e.,*

$$\{S \subset \mathbb{T}\colon\ S = (S^+)^-\} = \{S \subset \mathbb{T}\colon\ \exists\, F \in \mathcal{F} \text{ such that } S = F^+\}. \tag{3.6}$$

(3) *Both $F \mapsto F^+ = h(F)$ and $F \mapsto |F^+| = |h(F)|$ are continuous with respect to the Hausdorff metric.*

(4) *If $S \in \mathcal{S}$ and S^- is connected, then also S is connected.*

(5) *If $F \in \mathcal{F}$ is convex, then F^+ and F^- are convex and $F = (F^+)^-$. If $F_1, F_2 \in \mathcal{F}$ are convex and $F_1^+ = F_2^+$, then also $F_1 = F_2$.*

(6) *The set $\mathcal{S}$ is the closure in $\mathcal{F}$ of the set $\mathcal{S}^{fin} \subset \mathcal{S}$, where $\mathcal{S}^{fin}$ consists of all S of the form $S = h(\gamma)$ with $\gamma \subset \mathbb{T}$ finite.*

(7) *If $S \in \mathcal{S}$, then* $\mathrm{reach}(S^-) \geq 2$, *provided the following condition is satisfied:*

(C) S^- is connected and each component of $\mathbb{T} \setminus S^-$ contains exactly one component of $\mathbb{T} \setminus S$.

(8) *For any $S \in \mathcal{S}$ such that* $\mathrm{reach}(S^-) > 0$, *the boundary ∂S^- is 1-rectifiable. If $S \in \mathcal{S}^{fin}$, then the boundary ∂S^- is Lipschitz.*

Proof. We indicate the proper references to the literature. Part (7) is delicate.

(1) (Matheron [28, Chapter 1.5]) Note that $x \in (F^-)^+$ is equivalent to $B_2(x) \cap F^- \neq \emptyset$, which is equivalent to the existence of a $z \in B_2(x) \cap F^-$. Hence, $x \in B_2(z)$ since $z \in B_2(x)$ and $B_2(z) \subset F$ since $z \in F^-$. In [28], sets F such that $(F^-)^+ = F$ are called open w.r.t. $B_2(0)$ (rather than admissible).

(2) (Matheron [28, Chapter 1.5]). For any $S \in \mathcal{S}$, we have $S = F^+$ with $F = S^-$. On the other hand, if $S = F^+$, then $((F^+)^-)^+ \supset F^+$ since $(F^+)^- \supset F$ and hence $(S^-)^+ \supset S$. The inclusion $((F^+)^-)^+ \subset F^+$, which amounts to $(S^-)^+ \subset S$, was proven in 1.

(3) For the first claim, see Matheron [28, Proposition 1.5.1] Schneider and Weil [32, Theorem 12.3.5], for the second claim, see Kampf [25, Lemma 9].

(4) See Matheron [28, Chapter 1.5].

(5) See Matheron [28, Proposition 1.5.3]. In [28] , sets F such that $(F^+)^- = F$ are called closed w.r.t. $B_2(0)$.

(6) This follows from the fact that finite sets are dense in $\mathcal{F}$ in combination with the first claim in (3).

(7) Use Federer [14, Theorem 5.9], which assures that the reach is conserved under taking limits of sets with respect to the Hausdorff metric. It therefore suffices to consider $S \in \mathcal{S}^{\text{fin}}$ with $S = h(\gamma)$, where the finite set γ is sufficiently dense so that condition (C) is satisfied for $S = h(\gamma)$.

We will prove that, for such γ, $\text{reach}(h(\gamma)^-) \geq 2$. First, observe that the boundaries $\partial h(\gamma)$ and $\partial h(\gamma)^-$ are unions of circular arcs (of radius 2). Given that $h(\gamma)$ satisfies condition (C), the set $h(\gamma) \setminus h(\gamma)^-$ splits into connected components, each bordered by two Jordan curves: one connected component of the boundary $\partial h(\gamma)$ and one connected component of the boundary $\partial h(\gamma)^-$. For each component of $\partial h(\gamma)$, we label the centres of the arc circles in such a way that two consecutive arcs belong to two consecutive centres, with periodic boundary condition. The associated component of $\partial h(\gamma)^-$ is a union of circular arcs passing through the centres. The centre of the arc circle connecting two consecutive centres is the point on the boundary $\partial h(\gamma)$ in the intersection of the arcs with these centres.

Let us assume that $\text{reach}(h(\gamma)^-) < 2$. Then there exist a point $x \in h(\gamma) \setminus h(\gamma)^-$ and two distinct points y_1, y_2 in a connected component σ of the boundary $\partial h(\gamma)^-$ such that $\text{dist}(x, h(\gamma)^-) = \text{dist}(x, y_1) = \text{dist}(x, y_2) = r < 2$. (To belong to two distinct components of $\partial h(\gamma)^-$ contradicts assumption (C).) Given that $\text{dist}(x, h(\gamma)^-) = r$, the interior $B_r(x)^0$ of the disc $B_r(x)$ does not contain any point from $h(\gamma)^-$, i.e., $B_r(x)^0 \cap h(\gamma)^- = \emptyset$. In addition, there are at most finitely many points of $h(\gamma)^-$ in $\partial B_r(x)$, all of which belong to $\partial h(\gamma)^-$.

The admissibility of $h(\gamma)$ means that every disc of radius 2 with a centre on $\partial h(\gamma)^-$ is fully contained in $h(\gamma)$. We will draw a contradiction with this statement from the fact that a Jordan curve σ avoiding $B_r(x)^0$ and containing two distinct points on its boundary necessarily indents too sharply to be consistent with admissibility of $h(\gamma)$ and condition (C). To show the contradiction, consider a line ℓ through the point x that separates the points y_1 and y_2 into opposite half planes determined by ℓ. Without loss of generality, we may assume that the points $\{y, y'\} = \ell \cap \partial B_r(x)$ do not belong to $h(\gamma)^-$. As a result, both $B_2(y)$ and $B_2(y')$ contain points not in $h(\gamma)$ or, since $B_2(y_1) \cup B_2(y_2) \subset h(\gamma)$, there exist points $w \in B_2(y) \setminus (B_2(y_1) \cup B_2(y_2))$ and $w' \in B_2(y') \setminus (B_2(y_1) \cup B_2(y_2))$ such that $w, w' \notin h(\gamma)$. Now, $B_2(w)$ and $B_2(w')$ cannot contain any point from $h(\gamma)^-$, and hence the Jordan curve σ must avoid $B_r(x)^0 \cup B_2(w) \cup B_2(w') \cup \{y, y'\}$ with $w, w' \notin h(\gamma)$, which yields the contradiction with condition (C).

Indeed, if σ is the outer boundary of the set $h(\gamma)^-$ with a single outer component of $\mathbb{T} \setminus h(\gamma)^-$, then in contradiction with condition (C) this component contains two components of $\mathbb{T} \setminus h(\gamma)$, since the points w and w' belong to different components of $\mathbb{T} \setminus (h(\gamma)^- \cup B_2(y_1) \cup B_2(y_2))$. Otherwise, the Jordan curve σ is the inner boundary of the set $h(\gamma)^-$ and surrounds the set $B_r(x)^0 \cup B_2(w) \cup B_2(w') \cup \{y, y'\}$. However, the region encircled by σ contains two different components of $\mathbb{T} \setminus h(\gamma)$, one containing w and the other containing w'.

(8) For the first claim, see Ambrosio, Colesanti and Villa [2, Proposition 3]. For the second claim, it suffices to note that for $S \in \mathcal{S}^{\text{fin}}$ the boundary ∂S^- is a finite union of arcs. $\square$

We will also need the Steiner formula for sets of positive reach (Federer [14]).

Lemma 3.2. *Let $S \in \mathcal{S}$ be an admissible set with S^- of reach at least 2. Then*

$$|S \setminus S^-| = 2\,\mathcal{SM}(S^-) + 4\pi\chi(S^-), \tag{3.7}$$

where $\mathcal{SM}(S^-)$ is the outer Minkowski content of S^- and $\chi(S^-)$ is the Euler-Poincaré characteristic of S^- (= the number of connected components minus the number of holes). If the boundary ∂S^- is Lipschitz, then $\mathcal{SM}(S^-) = \mathcal{H}^1(\partial S^-)$, where $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure.

Proof. Reformulating the Steiner formula for sets of positive reach as defined by Federer [14, Theorem 5.5, Theorem 5.19], we get, for $S \subset \mathbb{R}^2$ and $S \in \mathcal{S}$,

$$|S^- + B_r(0)| = |S^-| + \mathcal{SM}(S^-)r + \chi(S^-)|B_1(0)|r^2 \tag{3.8}$$

for any $0 < r < 2$ and by continuity also for $r = 2$. For continuity of the left-hand side, see Sz.-Nagy [29]. The last claim is the same as Ambrosio, Colesanti and Villa [2, Corollary 1]. □

3.2 Minimisers of the shape rate function and their stability

In this section we prove Theorem 2.2.

(1) The proof relies on the Brunn-Minkowski inequality and on Lemma 3.3 below, which provides three reformulations of the isoperimetric inequality in (2.6).

Lemma 3.3. *Let $S \in \mathcal{F}$. If $R > 2$ and $|S| = \pi R^2$, then the following three statements are equivalent:*

(a) $|S| - \kappa|S^-| \geq \pi R^2 - \kappa\pi(R-2)^2$.

(b) $16\pi|S| \leq (|S \setminus S^-| + 4\pi)^2$.

(c) $16\pi|S^-| \leq (|S \setminus S^-| - 4\pi)^2$.

Moreover, equality holds in (a), (b), and (c) simultaneously, or in none.

Proof. The equivalence of (b) and (c) is an immediate consequence of the fact that $|S| = |S^-| + |S \setminus S^-|$. For the equivalence of (a) and (b), we observe that $|S| - \kappa|S^-| = \kappa|S \setminus S^-| - (\kappa - 1)|S|$ and therefore, with $|S| = \pi R^2$, (a) is equivalent to

$$|S \setminus S^-| \geq \pi R^2 - \pi(R-2)^2 = 4\pi R - 4\pi. \tag{3.9}$$

We add 4π to both sides and take the square to find that (a) is equivalent to (b). □

Proof of Theorem 2.2. Armed with Lemma 3.3, we employ the Brunn-Minkowski inequality

$$|F + B|^{1/2} \geq |F|^{1/2} + |B|^{1/2}, \tag{3.10}$$

which is valid for any non-empty measurable F, B and $F + B$ (see Lusternik [27] and Federer [15, 3.2.41]). Indeed, (3.10) with $B = B_2(0)$ implies

$$|F^+| - |F| \geq 2|F|^{1/2}(4\pi)^{1/2} + 4\pi, \tag{3.11}$$

and yields inequality (c) with $F = S^-$ and $F^+ = S$, and thus also (2.6) by (a). Equality in (3.11) occurs only if $F = S^-$ is a disc or a point (see Burago and Zalgaller [3, Section 8.2.1]).

(2) We first prove that if S is close to a minimiser, then necessarily S^- is connected and simply connected.

Lemma 3.4. *There exist a function $\varepsilon \mapsto \xi(\varepsilon)$ satisfying $\lim_{\varepsilon \downarrow 0} \xi(\varepsilon) = 0$ such that if $S \in \mathcal{S}$ satisfies (2.7) with $R - 2 \geq \frac{\xi(\varepsilon)}{1-\xi(\varepsilon)}$, then*

$$d_{\mathrm{H}}(S, B_R) \leq \xi(\varepsilon) \tag{3.12}$$

for sufficiently small ε, and S^- is connected and simply connected.

Proof. We use the claim about the stability of the Brunn-Minkowski inequality, first proven in qualitatively by Christ [6] and then quantitatively by Figalli and Jerrison [17]. Actually, for our purposes the qualitative version [6] suffices, since we are only using it as a springboard for more accurate bounds to be considered later.

Adapted to our setting, the claim is that there exist positive function $\widetilde{\xi}(\delta)$ with $\lim_{\delta\to 0}\widetilde{\xi}(\delta)=0$, such that if

$$|S|^{1/2} \le |S^-|^{1/2} + 2\sqrt{\pi} + \delta \max\left(|S^-|^{1/2}, 2\sqrt{\pi}\,\right), \tag{3.13}$$

with sufficiently small δ, then there exist a compact convex set $K \subset \mathbb{T}$ such that

$$S^- \subset K^-, \qquad |K^- \setminus S^-| < \pi R^2 \tilde{\xi}(\delta). \tag{3.14}$$

In the same vein, a (properly scaled and shifted) disc D satisfies

$$D^- \subset K^-, \qquad |K^- \setminus D^-| < \pi R^2 \tilde{\xi}(\delta). \tag{3.15}$$

To verify the assumption in (3.13), we rewrite (2.7) as

$$\big(|S| - \kappa|S^-|\big) - \big(\pi R^2 - \kappa\pi(R-2)^2\big) = \kappa\big(|S\setminus S^-| - 4\pi(R-1)\big) \le \pi\kappa\varepsilon \tag{3.16}$$

and use an equivalent formulation of (3.13), namely,

$$|S\setminus S^-| - 4\sqrt{\pi}\,|S|^{1/2} + 4\pi \le 2|S^-|^{1/2}\delta \max\left(|S^-|^{1/2}, 2\sqrt{\pi}\,\right) + \delta^2 \max(|S^-|, 4\pi). \tag{3.17}$$

Indeed, (3.16) with $|S| = \pi R^2$ implies that the LHS of (3.17) is bounded by $\pi\varepsilon$ which, with the choice $\delta = \sqrt{\varepsilon}$ is bounded by the right-hand side of (3.17) and thus also (3.13).

Note that, in view of the convexity of K^-, the condition in (3.15) implies that $K^- \subset (D^- + B_{\tilde{\xi}(\sqrt{\varepsilon}\,)^2}(0))$. Indeed, suppose without loss of generality that the centre of the disc D is at the origin and write $r+2$ for its radius, so that $D^- = B_r(0)$. If $x \in K^- \setminus D^-$, then K^- contains the union of D^- and the wedge bordered by ∂D^- and the tangents through x to D^-. The volume of this wedge is $r(\sqrt{x^2-r^2} - \arctan(\frac{\sqrt{x^2-r^2}}{r}))$. Asymptotically, for small $|x|-r$, this equals $\frac{1+r^2}{\sqrt{r}}\sqrt{|x|-r}$ and exceeds $\pi R^2\tilde{\xi}$ once $|x| > r(1+\tilde{\xi}^2)$. Thus, $K^- \subset B_{r+\tilde{\xi}}(0)$ and

$$S \subset K \subset B_{r+2+r\tilde{\xi}^2}(0). \tag{3.18}$$

For an admissible $S \in \mathcal{S}$, we can significantly strengthen the second claim in (3.14). We can argue that $S \supset B_{r+2-s}(0) = B_{r+2}(0) \ominus B_s(0)$ with $s = (\frac{1}{2}\pi R^2\tilde{\xi}(\sqrt{\varepsilon}\,))^{2/3}$. Indeed, if $x \in (\mathbb{T}\setminus S)\cap B_{r+2-s}(0)$, then $B_2(x)\cap B_{r-s}(0)\cap S^- = \emptyset$, while $|B_2(x)\cap K^-| \ge |B_2(x)\cap B_r(0)| \ge \frac{8}{3}s^{3/2} = \frac{4}{3}\pi R^2\tilde{\xi}(\sqrt{\varepsilon}\,)$ (see [22, Eq. (D.8)], in contradiction with the second inequality in (3.14). Combining the present claim with (3.18), we get

$$B_{r+2-s}(0) \subset S \subset B_{r+2+r\tilde{\xi}^2}(0). \tag{3.19}$$

Given that $|S| = \pi R^2$, this implies

$$\frac{R + 2\tilde{\xi}^2}{1+\tilde{\xi}^2} \le r+2 \le R+s, \tag{3.20}$$

yielding, for sufficiently small ε, the statement in (3.12) with

$$\xi(\varepsilon) = 2R^{4/3}\tilde{\xi}^{2/3}(\sqrt{\varepsilon}\,). \tag{3.21}$$

Using that

$$B_{R-2-\xi}(0) \subset S^- \subset B_{R-2+\xi}(0), \tag{3.22}$$

we will prove connectedness and simple connectedness of the set S^- by showing that every segment $\ell_e = \{te\colon\ t\in[0,R+\xi]\}$ in the direction of any unit vector $e\in\mathbb{R}^2$, intersects the set S^- in a closed interval: $S^-\cap\ell_e = \{te\colon\ t\in[0,T(e)]\}$ with $T(e)\in[R-2-\xi, R-2+\xi]$. If the contrary were true, then there would be a direction e and two points $x,y\in\{te\colon\ t\in[R-2-\xi,R-2+\xi]\}\subset\ell_e$, $|x|<|y|$ such that $x \notin S^-$ and $y \in S^-$. The fact that $y \in S^-$ implies that $B_{R-\xi} \cup B_2(y) \subset S$, while $x \notin S^-$ implies that $B_2(x) \cap S^c \neq \emptyset$, and hence, in view of the preceding inclusion, $B_2(x) \cap (B_{R-\xi} \cup B_2(y))^c \neq \emptyset$. However, this cannot happen when $B_2(x) \setminus B_2(y) \subset B_{R-\xi}$. Nevertheless, this is exactly what happens when $R-2 \geq \xi/(\xi-1)$. Indeed, $B_2(x) \setminus B_2(y) \subset B_{R-\xi}$ once $\partial B_2(y) \cap \ell \subset B_{R-\xi}(0)$, where ℓ is the line through y orthogonal to ℓ_e. For $z \in \partial B_2(y) \cap \ell$ we have

$$z^2 = y^2 + 2^2 \leq (R-2+\xi)^2 + 4 \leq (R-\xi)^2 \tag{3.23}$$

when $\frac{R-2}{R-1} \geq \xi$, which is equivalent with $R-2 \geq \frac{\xi}{1-\xi}$. □

To finish the proof of Theorem 2.2, we use that S^- is connected and simply connected once (2.7) is satisfied and $R-2 \geq \frac{\xi(\varepsilon)}{1-\xi(\varepsilon)}$. For the latter, similarly as in the proof of Lemma 3.1 7, we may assume that $S \in S^{\text{fin}}$. For fixed $R \in (2, \frac{1}{2}L)$ we choose ε_0 such that $R-2 \geq \frac{\xi(\varepsilon)}{1-\xi(\varepsilon)}$ for any $0 < \varepsilon \leq \varepsilon_0$. Using now that $\text{reach}(S^-) \geq 2$ according to Lemma 3.1(7), we will rely on the Bonnesen inequality, which is more precise than the provisional claim in (3.12) with the bound $\xi(\varepsilon)$ whose dependence on ε is not explicitly specified. For connected and simply connected S^-, its boundary ∂S^- is a Jordan curve and according to the Bonnesen inequality the difference of the radii $r_{\text{out}}(\partial S^-)$ and $r_{\text{in}}(\partial S^-)$ of the outer and the inner circle of the curve ∂S^- can be bounded in terms of the isoperimetric defect $\mathcal{H}^1(\partial S^-)^2 - 4\pi|S^-|$:

$$\pi^2\big(r_{\text{out}}(\partial S^-) - r_{\text{in}}(\partial S^-)\big)^2 \leq \mathcal{H}^1(\partial S^-)^2 - 4\pi|S^-|. \tag{3.24}$$

Assuming that $|S| = \pi R^2$, we can write

$$\begin{aligned}&\big(|S| - \kappa|S^-|\big) - \big(\pi R^2 - \kappa\pi(R-2)^2\big)\\ &\quad= \kappa|S \setminus S^-| - (\kappa-1)|S| - \pi R^2 + \kappa\pi(R-2)^2 = 2\kappa\big[\mathcal{H}^1(\partial S^-) - 2\pi(R-2)\big]\end{aligned} \tag{3.25}$$

with the help of the Steiner formula (Lemma 3.2) and reformulate the isoperimetric defect as

$$\mathcal{H}^1(\partial S^-)^2 - 4\pi|S^-| = \mathcal{H}^1(\partial S^-)^2 - 4\pi[|S| - 4\pi - 2\mathcal{H}^1(\partial S^-)] = (\mathcal{H}^1(\partial S^-) + 4\pi)^2 - (2\pi R)^2. \tag{3.26}$$

If the left-hand side of (3.25) is $\leq \pi\kappa\varepsilon$, then the right-hand side of (3.26) is bounded from above by

$$\tfrac{\pi\varepsilon}{2}\left(\tfrac{\pi\varepsilon}{2} + 4\pi R\right) = \pi^2\varepsilon\,(2R + \tfrac{\varepsilon}{4}) < \tfrac{9}{4}\pi^2 R\varepsilon. \tag{3.27}$$

It follows from (3.24) that

$$r_{\text{out}}(\partial S^-) - r_{\text{in}}(\partial S^-) \leq \tfrac{3}{2}\sqrt{R\varepsilon}. \tag{3.28}$$

Since $r_{\text{out}}(\partial S) - r_{\text{in}}(\partial S) = r_{\text{out}}(\partial S^-) - r_{\text{in}}(\partial S^-)$, we get the claim in (2.8). □

3.3 Large deviation principle for Widom-Rowlinson

In this section we prove Theorems 2.1 and 2.3.

Remember the Gibbs measure μ_β at inverse temperature β and activity $z = \kappa\, z_t(\beta)$. This is a probability measure on the space Γ of particle configurations, which we may view as a subset of $\mathcal{F}$ equipped with the Hausdorff topology. By a slight abuse of notation, we identify μ_β on Γ with the measure on $\mathcal{F}$ supported on Γ. Theorem 2.1 builds on the large deviation principle for the Gibbs measure μ_β itself, i.e., the large deviation principle for the set of particle locations.

The LDP for μ_β is summarised in the following proposition (recall that a rate function is called good when it is lower semi-continuous and has compact level sets). This proposition is in the spirit of Schreiber [33], [34, Theorem 1]. The latter is stated in a slightly different setting, but the main ideas of the proof carry over.

Proposition 3.5 (Large deviation principle for Widom-Rowlinson)**.** *The family of probability measures $(\mu_\beta)_{\beta\geq 1}$ on $\mathcal{F}$, supported on $\Gamma \subset \mathcal{F}$, satisfies the LDP with rate β and good rate function I^{WR} given by*

$$I^{\text{WR}} = J^{\text{WR}} - \inf_{\mathcal{F}} J^{\text{WR}}, \quad J^{\text{WR}}(F) = |F^+| - \kappa|F|, \quad F \in \mathcal{F}. \tag{3.29}$$

Proof. Let $\Pi_{\kappa\beta}$ be the homogeneous Poisson point process on $\mathbb{T}$ with intensity $\kappa\beta$. We may view $\Pi_{\kappa\beta}$ as a random variable on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ taking values in $\mathcal{F}$. Since $\mathbb{P}(\Pi_{\kappa\beta} \subset F) = \mathbb{P}(\Pi_{\kappa\beta} \cap (\mathbb{T}\backslash F) = \emptyset) = \mathrm{e}^{-\kappa\beta|\mathbb{T}\backslash F|}$, $F \in \mathcal{F}$, it follows that the family $(\Pi_{\kappa\beta})_{\beta\geq 1}$ satisfies the large deviation principle with rate β and good rate function $I(F) = \kappa|\mathbb{T}\backslash F|$, $F \in \mathcal{F}$. Note that $F \mapsto |F|$ is upper semi-continuous (Schneider and Weil [32, Theorem 12.3.5]), but not continuous: sets F of positive measure can be approximated by finite sets, which have measure zero. It follows that $F \mapsto I(F) = \kappa(|\mathbb{T}| - |F|)$ is lower semi-continuous, but not continuous. Nevertheless, the map $F \mapsto |F^+| = |h(F)|$ is continuous with respect to the Hausdorff metric (see Lemma 3.1(3)). Therefore, since

$$\mu_\beta(\mathcal{C}) = \mathrm{e}^{-|\mathbb{T}|}\, \mathrm{e}^{\kappa\beta|\mathbb{T}|} \mathbb{E}\left(\mathrm{e}^{-\beta|h(\Pi_{\kappa\beta})|} \mathbf{1}_{\{\Pi_{\kappa\beta}\in\mathcal{C}\}}\right), \qquad \mathcal{C} \subset \mathcal{F} \text{ Borel}, \tag{3.30}$$

the claim follows from the LDP for the Poisson point process $(\Pi_{\kappa\beta})_{\beta\geq 1}$ and Varadhan's lemma. $\square$

With the help of Proposition 3.5, the proof of Theorems 2.1 and 2.3 becomes straightforward via the contraction principle.

Proof of Theorem 2.1. As mentioned above, the map $\mathcal{F} \to \mathcal{S}$ defined by $F \mapsto S = F^+$ is continuous with respect to the Hausdorff metric. Proposition 3.5 and the contraction principle therefore imply that the LDP for the law of $h(\gamma)$ under μ_β holds with rate β and good rate function I given by

$$I = J - \inf_{\mathcal{F}} J, \quad J(S) = \inf\left\{|F^+| - \kappa|F| \colon\ F \in \mathcal{F}, F^+ = S\right\}, \quad S \in \mathcal{S}. \tag{3.31}$$

We show that $J(S) = |S| - \kappa|S^-|$. Indeed, if $F^+ = S$, then $F \subset S^-$ and $|F^+| - \kappa|F| \geq |S| - \kappa|S^-|$ yielding $J(S) \geq |S| - \kappa|S^-|$. On the other hand, taking $F = S^-$ with $F^+ = (S^-)^+ = S$ in view of admissibility of S, we get $J(S) \leq |F^+| - \kappa|F| = |S| - \kappa|S^-|$. $\square$

Proof of Theorem 2.3. Theorem 2.3 follows from Proposition 3.5, the continuity of the map $F \mapsto |F^+| = |h(F)|$, and the contraction principle. The rate function I^* is given by

$$\begin{aligned} I^*(A) &= \inf\{I(S)\colon\ S \in \mathcal{S},\, |S| = A\} \\ &= \inf\{|S| - \kappa|S^-|\colon\ S \in \mathcal{S},\, |S| = A\} - \inf\{|S| - \kappa|S^-|\colon\ S \in \mathcal{S}\}. \end{aligned} \tag{3.32}$$

In the difference of the two infima, the first infimum (when $A = \pi R^2$ with $R \in (2, \frac{1}{2}L)$) is equal to $\pi R^2 - \kappa\pi(R-2)^2$ by Theorem 2.2. For the second infimum, we note that

$$|S| - \kappa|S^-| \geq (1-\kappa)|S| \geq (1-\kappa)|\mathbb{T}| \tag{3.33}$$

with equality for $S = \mathbb{T}$. $\square$

4 Heuristics for moderate deviations

In this section we provide the main ideas behind the proof of Theorems 2.5–2.6 in Sections 5–8. Guidance is needed because the proof is long and intricate. In Section 4.1 we explain how the moderate deviation probability for the halo volume can be expressed in terms of a certain *surface integral*. In Section 4.2 we explain how the weight in this surface integral can be approximated in terms of the *polar coordinates* of the boundary points. In Section 4.3 we provide a quick guess of what the *orders of magnitude* of the angles and the radii of the boundary points are as $\beta \to \infty$. In Section 4.4 we introduce *auxiliary random processes* that allow us to transform the surface integral into an expectation of a certain exponential functional, capturing the *global* (= mesoscopic) scaling of the boundary of the critical droplet. In Section 4.5 we perform a further change of variable to rewrite the expectation in terms of an *effective interface model*, capturing the *local* (= microscopic) scaling of the boundary of the critical droplet.

4.1 Reduction to a surface integral

The starting point for the proof of Theorem 2.5 is the following. Because of the large deviation principles in Theorems 2.1 and 2.3 and the quantitative isoperimetric inequality 2.2, the dominant contribution to the event $|V(\gamma) - \pi R_c^2| \le C\beta^{-2/3}$ should come from approximately disc-shaped halos ("droplets"). Consider the event

$$\mathcal{D}_\varepsilon(x) := \Big\{\gamma \in \Gamma\colon\ d_{\mathrm{H}}(\partial h(\gamma), \partial B_{R_c}(x)) \le \varepsilon\Big\} \tag{4.1}$$

that $h(\gamma)$ is close to a disc $B_{R_c}(x)$, without any holes. Because of the translation invariance of the model, we may focus on $\mathcal{D}_\varepsilon(0)$.

For $\gamma \in \mathcal{D}_\varepsilon(0)$, the boundary $\partial h(\gamma)$ of the droplet is a union of circle arcs centreed at points $z_1, \dots, z_n \in \gamma$, called *boundary* points. Each boundary point is *extremal* in the sense that $h(\gamma \setminus x) \subsetneq h(\gamma)$. We call a collection of points $\{z_1, \dots, z_n\}$ a *connected outer contour* if there exists a halo S with a simply connected 2-interior S^- having exactly these boundary points. The halo S, if it exists, is unique, and we denote it by $S(z)$. The set of connected outer contours is denoted by $\mathcal{O}$.

For $\gamma \in \mathcal{D}_\varepsilon(0)$, both $h(\gamma)$ and $V(\gamma)$ are uniquely determined by the boundary points, since $h(\gamma) = S(z)$ and $V(\gamma) = S(z)^-$. Therefore it makes sense to compute probabilities by conditioning on the boundary points. Abbreviate

$$\Delta(z) = \big(|S(z)| - \kappa |S(z)^-|\big) - \big(\pi R_c^2 - \kappa (R_c - 2)^2\big). \tag{4.2}$$

We will see that the following is true: for some constant $c > 0$ and for all measurable sets $A \subset \mathcal{S}$, as $\beta \to \infty$,

$$\begin{aligned} &\mu_\beta\Big(h(\gamma) \in A,\, \gamma \in \mathcal{D}_\varepsilon(0)\Big) \\ &\qquad = \big[1 - O(\mathrm{e}^{-c\beta})\big]\, e^{-\beta I^*(\pi R_c^2)} \sum_{n \in \mathbb{N}_0} \frac{(\kappa\beta)^n}{n!} \int_{\mathbb{T}^n} \mathrm{d}z\, \mathrm{e}^{-\beta \Delta(z)}\, 1\!\!1_{\{S(z) \in A\}}\, 1\!\!1_{\mathcal{D}'_\varepsilon(0)}(z), \end{aligned} \tag{4.3}$$

where $\mathcal{D}'_\varepsilon(0)$ is the collection of connected outer contours for which $d_{\mathrm{H}}(\partial S(z), \partial B_{R_c}(0)) \le \varepsilon$, and I^* is the rate function defined in (2.10). In view of this geometric constraint, the only contributions to the right-hand side of (4.3) are from connected outer contours $z \in \mathcal{O}$ that lie in an annulus: $|z_i - (R_c - 2)| \le \varepsilon$, $1 \le i \le n$. Hence we may think of (4.3) as a *surface integral.*

4.2 Approximation of the surface term

In view of (4.3), our next task is to evaluate $\Delta(z)$. Let us choose polar coordinates for the boundary points and write

$$z_i = (r_i \cos t_i, r_i \sin t_i), \qquad 1 \le i \le n. \tag{4.4}$$

Upon relabelling the centres, we may without loss of generality assume that $0 \le t_1 \le \cdots \le t_n < 2\pi$. We set $t_{n+1} = t_0 + 2\pi$ and $r_{n+1} = r_1$, and define angular increments

$$\theta_i = t_{i+1} - t_i, \qquad 1 \le i \le n. \tag{4.5}$$

Note that $\theta_i \ge 0$ and $\sum_{i=1}^n \theta_i = 2\pi$. The volume of the shape $S(z)$ with boundary points $z = (z_1, \dots, z_n)$ admits an expansion

$$|S(z)| = \sum_{i=1}^n \Big\{\frac{1}{2}\Big(\frac{r_i + r_{i+1}}{2} + 2\Big)^2 \theta_i + \frac{(r_{i+1} - r_i)^2}{(R-2)\theta_i} - \frac{R^2(R-2)}{48}\,\theta_i^3\Big\} + \text{error terms}. \tag{4.6}$$

Also,

$$|S(z)^-| = \sum_{i=1}^n \Big\{\frac{1}{2}\Big(\frac{r_i + r_{i+1}}{2}\Big)^2 \theta_i - \frac{R(R-2)^2}{24}\,\theta_i^3\Big\} + \text{error terms}. \tag{4.7}$$

The previous formulas are valid for general radius R. Next we specialize to $R = R_c$. With

$$\rho_i = r_i - (R_c - 2), \qquad \bar\rho_i = \frac{\rho_i + \rho_{i+1}}{2}, \tag{4.8}$$

and using that $\kappa - 1 = 2/(R_c - 2)$, we get

$$\Delta(z) = \frac{\kappa-1}{2}\sum_{i=1}^{n}\Big\{\frac{(\rho_{i+1}-\rho_i)^2}{\theta_i} - \bar\rho_i^2\theta_i\Big\} + C_1\sum_{i=1}^{n}\theta_i^3 + \text{error terms} \tag{4.9}$$

with

$$C_1 = \frac{R_c^2(R_c-2)}{48} = \frac{\kappa^2}{6(\kappa-1)^3} = \frac{G_\kappa^3}{24}. \tag{4.10}$$

4.3 Orders of magnitude

Neglecting higher order terms in $\Delta(z)$ in (4.9), we see that the weight $\exp(-\beta\Delta(z))$ involves several terms. The factor $\exp(-\beta C_1\theta_i^3)$ suggests that the typical angular increment θ_i is of order $\beta^{-1/3}$, and the typical number of boundary points therefore is of order $\beta^{1/3}$. The factor

$$\exp\Big(-\beta\frac{\kappa-1}{2}\frac{(\rho_{i+1}-\rho_i)^2}{\theta_i}\Big) \tag{4.11}$$

suggests that $\rho_{i+1} - \rho_i$ is approximately normal with variance proportional to θ_i/β. Hence, we expect that the radial increment $\rho_{i+1} - \rho_i$ is typically of order $\beta^{-2/3}$. Combining these observations, we may expect that

$$\beta\sum_{i=1}^{n}\Big\{\frac{\kappa-1}{2}\frac{(\rho_{i+1}-\rho_i)^2}{\theta_i} + C_1\theta_i^3\Big\} \approx \text{const}\,\beta^{1/3}, \tag{4.12}$$

which explains the exponent $\beta^{1/3}$ in Theorem 2.5. Furthermore, it will be natural to think of ρ_i as

$$\rho_i = \frac{m + B_{t_i}}{\sqrt{(\kappa-1)\beta}} \tag{4.13}$$

with m some unknown mean value, and $(B_t)_{t\geq 0}$ the mean-centred Brownian bridge from (2.18). The consistency with the guessed order of magnitude of the radial increment is guaranteed by the fact that $B_{t_{i+1}} - B_{t_i} \approx \sqrt{\theta_i} \approx \beta^{-1/6}$ and the observation that $\beta^{-1/2-1/6} = \beta^{-2/3}$. Finally, we note that

$$\beta\frac{\kappa-1}{2}\sum_{i=1}^{n}\bar\rho_i^2\theta_i \approx \pi m^2 + \frac{1}{2}\int_0^{2\pi} B_t^2\mathrm{d}t, \tag{4.14}$$

which should not contribute at the scale $\beta^{1/3}$ we are interested in (unless m is large). Nevertheless, we will need to treat this term carefully because, for the mean-centred Brownian bridge we have

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_0^{2\pi} B_t^2\mathrm{d}t\Big)\Big] = \infty, \tag{4.15}$$

and extra arguments will be needed to cure this divergence.

For later purpose, let us also have a closer look at the volume constraint $|V(\gamma) - \pi R_c^2| \leq C\beta^{-2/3}$. If we substitute the expression (4.6) for $V(\gamma)$ and neglect higher order terms, then the volume constraint becomes

$$\Big|\sum_{i=1}^{n}\Big\{\frac{1}{2}\big[(R_c+\bar\rho_i)^2 - R_c^2\big]\theta_i + \frac{(\rho_{i+1}-\rho_i)^2}{(R_c-2)\theta_i} - C_1\theta_i^3\Big\}\Big| \lesssim C\beta^{-2/3}. \tag{4.16}$$

Making a few leaps of faith, we may approximate

$$\begin{aligned}\sum_{i=1}^{n}\tfrac{1}{2}\big[(R_c+\bar\rho_i)^2 - R_c^2\big]\theta_i &\approx \frac{1}{2}\int_0^{2\pi}\Big(\big(R_c + [(\kappa-1)\beta]^{-1/2}(m+B_t)\big)^2 - R_c^2\Big)\mathrm{d}t\\ &= \pi\Big(\big(R_c + [(\kappa-1)\beta]^{-1/2}m\big)^2 - R_c^2\Big) + \frac{1}{2(\kappa-1)\beta}\int_0^{2\pi} B_t^2\mathrm{d}t,\end{aligned} \tag{4.17}$$

using that $\int_0^{2\pi} B_t \mathrm{d}t = 0$ for the mean-centred Brownian bridge. From the considerations above we should expect the sum overall to be of order $\beta^{-2/3}$. Hence $[(\kappa-1)\beta]^{-1/2} m$ should also be of order $\beta^{-2/3}$, i.e., $|m| = O(\beta^{-1/6})$. Later we will only prove that $|m| = O(\beta^{1/6})$, but this will turn out to be enough for our purpose.

4.4 Global scaling: auxiliary random processes

If we substitute the approximation (4.9) for $\Delta(z)$ into the surface integral in (4.3) and drop error terms and indicators, we are naturally led to the investigation of expressions of the type

$$\sum_{n\in\mathbb{N}_0} (\kappa\beta)^n \int_{[0,2\pi)^n} \mathrm{d}t\ 1_{\{t_1\le\cdots\le t_n\}} \int_{\mathbb{R}^n} \mathrm{d}\rho \prod_{i=1}^n (R_c - 2 + \rho_i)^n \\ \times \exp\Big(-\beta(\kappa-1)\Big\{\sum_{i=1}^n \frac{(\rho_{i+1}-\rho_i)^2}{2\theta_i} - \sum_{i=1}^n \tfrac12 \bar\rho_i^2\theta_i\Big\} - \beta C_1 \sum_{i=1}^n \theta_i^3\Big) f\big(\{z_i\}_{i=1}^n\big), \tag{4.18}$$

where f is some non-negative test function, and we recall (4.4) and (4.8) (by convention the summand with $n = 0$ equals 1). The Gaussian term (see also (4.11)) is conveniently expressed with the heat kernel

$$P_\theta(x-y) = \frac{1}{\sqrt{2\pi\theta}} \exp\Big(-\frac{(x-y)^2}{2\theta}\Big), \tag{4.19}$$

and we have

$$\begin{aligned} &\exp\Big(-\beta(\kappa-1)\sum_{i=1}^n \frac{(\rho_{i+1}-\rho_i)^2}{2\theta_i} - \beta C_1 \sum_{i=1}^n \theta_i^3\Big) \\ &\quad = \prod_{i=1}^n P_{\theta_i}\Big(\sqrt{(\kappa-1)\beta}(\rho_{i+1}-\rho_i)\Big) \times \prod_{i=1}^n \sqrt{2\pi\theta_i}\, \mathrm{e}^{-\beta C_1\theta_i^3}. \end{aligned} \tag{4.20}$$

Let us approximate $R_c - 2 + \rho_i \approx R_c - 2$, drop the term $\sum_i \bar\rho_i^2\theta_i$, and change variables as $x_i = \sqrt{(\kappa-1)\beta}\,\rho_i$. Then the integral in (4.18) becomes

$$\sum_{n\in\mathbb{N}_0} \Big(\frac{\kappa\beta(R_c-2)}{\sqrt{(\kappa-1)\beta}}\Big)^n \int_{[0,2\pi)^n} \mathrm{d}t\ 1_{\{t_1\le\cdots\le t_n\}} \int_{\mathbb{R}^n} \mathrm{d}x \prod_{i=1}^n \Big(P_{\theta_i}(x_{i+1}-x_i)\sqrt{2\pi\theta_i}\, \mathrm{e}^{-\beta C_1\theta_i^3}\Big) f\big(\{z_i\}_{i=1}^n\big). \tag{4.21}$$

With the help of

$$\frac{\kappa\beta(R_c-2)}{\sqrt{(\kappa-1)\beta}}\sqrt{2\pi\theta_i} = \frac{2\kappa\sqrt\beta}{(\kappa-1)^{3/2}}\sqrt{2\pi\theta_i} = \beta G_\kappa^{3/2}\sqrt{2\pi\theta_i} = \beta^{1/3} G_\kappa \sqrt{2\pi G_\kappa \beta^{1/3}\theta_i}, \tag{4.22}$$

and $C_1 = G_\kappa^3/24$, the expression (4.21) is rewritten as

$$\sum_{n\in\mathbb{N}_0} \big(\beta^{1/3}G_\kappa\big)^n \int_{[0,2\pi)^n} \mathrm{d}t\ 1_{\{t_1\le\cdots\le t_n\}} \int_{\mathbb{R}^n} \mathrm{d}x \prod_{i=1}^n \Big(P_{\theta_i}(x_{i+1}-x_i)\sqrt{2\pi\beta^{1/3}G_\kappa\theta_i}\, \mathrm{e}^{-\frac{1}{24}\beta G_\kappa^3\theta_i^3}\Big) f\big(\{z_i\}_{i=1}^n\big). \tag{4.23}$$

This expression motivates the auxiliary processes and the expressions Y_0 and Y_1 introduced before Theorem 2.5. Moreover, β and κ only enter in the combination $\beta^{1/3}G_\kappa$ (except possibly in the test function f), which explains the scaling of the surface corrections in Theorem 2.5.

The picture that emerges of the droplet boundary is that its deviation from $\partial B_{R_c}(0)$ should be of the order of $\beta^{-1/2}$, and that the boundary points are obtained by selecting points of a Gaussian curve according to an angular point process. The process $\mathcal{T}$ has a number of points of order $\beta^{1/3}$. We may call this picture *global* since it describes the overall shape of the droplet – the Gaussian process $(B_t)_{t\in[0,2\pi]}$ does not see the microscopic details.

4.5 Local scaling: effective interface model

Equation (4.18) suggests one last change of variables, namely, set

$$s_i = \beta^{1/3} G_\kappa t_i, \quad \varphi_i = \sqrt{\beta^{1/3} G_\kappa}\, x_i, \quad \vartheta_i = s_{i+1} - s_i = \beta^{1/3} G_\kappa\, \theta_i. \tag{4.24}$$

Note that

$$\rho_i = \frac{x_i}{\sqrt{(\kappa-1)\beta}} = \frac{\varphi_i}{\beta^{2/3}(2\kappa)^{2/3}}. \tag{4.25}$$

Then (4.21) becomes

$$\sum_{n\in\mathbb{N}_0} \int_{[0,2\pi G_\kappa \beta^{1/3})^n} \mathrm{d}s\ 1_{\{s_1\le\cdots\le s_n\}} \int_{\mathbb{R}^n} \mathrm{d}\varphi \prod_{i=1}^n \Big(P_{\vartheta_i}(\varphi_{i+1}-\varphi_i)\sqrt{2\pi\vartheta_i}\, \mathrm{e}^{-\frac{1}{24}\vartheta_i^3} \Big) f\big(\{z_i\}_{i=1}^n\big) \tag{4.26}$$

or, equivalently,

$$\sum_{n\in\mathbb{N}_0} \int_{[0,2\pi G_\kappa \beta^{1/3})^n} \mathrm{d}s\ 1_{\{s_1\le\cdots\le s_n\}} \int_{\mathbb{R}^n} \mathrm{d}\varphi\ \exp\left(-\sum_{i=1}^n \frac{(\varphi_{i+1}-\varphi_i)^2}{2\vartheta_i} - \sum_{i=1}^n \frac{\vartheta_i^3}{24} \right) f\big(\{z_i\}_{i=1}^n\big). \tag{4.27}$$

Let us finally return to the term $\sum_{i=1}^n \bar\rho_i^2 \theta_i$ that we had dropped from (4.18). By (4.25), we have

$$\beta\, \frac{\kappa-1}{2} \sum_{i=1}^n \bar\rho_i^2 \theta_i = \frac{1}{2G_\kappa^2 \beta^{2/3}} \sum_{i=1}^n \bar\varphi_i^2 \vartheta_i. \tag{4.28}$$

Taking this term into account, we see that (4.27) should be replaced by the more accurate integral

$$\int_{\mathbb{R}^n} \mathrm{d}\varphi \int_{[0,2\pi G_\kappa \beta^{1/3})^n} \mathrm{d}s\ 1_{\{s_1\le\cdots\le s_n\}}\ \exp\left(\frac{1}{2G_\kappa^2\beta^{2/3}} \sum_{i=1}^n \bar\varphi_i^2\vartheta_i - \sum_{i=1}^n \Big\{ \frac{(\varphi_{i+1}-\varphi_i)^2}{2\vartheta_i} + \frac{\vartheta_i^3}{24} \Big\} \right) f\big(\{z_i\}_{i=1}^n\big). \tag{4.29}$$

We may view the exponential, together with an additional indicator for the boundary points, as the Boltzmann weight for an effective interface model, which is studied in detail in [23]. The term $\frac{1}{2G_\kappa^2\beta^{2/3}} \sum_{i=1}^n \bar\varphi_i^2 \vartheta_i$ plays the role of a background potential.

5 Stochastic geometry: approximation of geometric functionals

This section collects a number of geometric facts that will be needed for the moderate deviations of the halo volume. In Section 5.1 we prove a number of *a priori estimates* on the radial and the angular coordinates of the *boundary points*, i.e., the centres of the discs that lie at the boundary of the critical droplet (Lemma 5.1, Proposition 5.2, Definition 5.3 and Corollary 5.4). These estimates play a crucial role for the arguments in Sections 6–8. In Section 5.2 we show that the set of boundary points allows for a local characterisation, in the sense that whether or not a 2-disc touches the boundary of the critical droplet only depends on the centre of the two neighbouring 2-discs (Definition 5.5, Lemma 5.6 and Proposition 5.7). In Section 5.3 we derive an approximation for the *volume* and the *surface* of halos that are close to a critical disc, in terms of certain sums involving the radial and the angular coordinates of the boundary points (Proposition 5.8). In Section 5.4 we do the same for the *geometric centre* of halos that are close to a critical disc (Proposition 5.9 and Corollary 5.10). The a priori estimates in Sections 5.1 allow us to control the approximations derived in Sections 5.3–5.4.

5.1 A priori estimates on boundary points

Theorem 2.2 can be applied to sets of the form $S = h(\gamma)$, the halo of the configuration γ. In particular, the condition that the boundary ∂S is close to a disc B_R with $R > 2$ is a strong restriction on the geometry of the boundary points $(z_1, \ldots, z_n)$ (recall Fig. 5).

In this section we collect several *a priori estimates* and constraints that follow from the fact that $S = h(\gamma)$ has a simply connected 2-interior S^- and $d_{\mathrm{H}}(\partial S, \partial B_R) \leq \varepsilon$. Remember the notion of boundary points, connected outer contour $\mathcal{O}$, and the polar introduced in Section 4.1 and the polar coordinates $(r_i, t_i)_{i=1}^n$ and angular increments θ_i from (4.4) and (4.5). We write ℓ_{z_i} or ℓ_i to denote the ray from the origin passing through the point z_i, and $A_{R,\varepsilon}$ and $A_{R-2,\ varepsilon}$ to denote the ε-annuli defined as the closures of $B_{R+\varepsilon}(0) \setminus B_{R-\varepsilon}(0)$ and $B_{R-2+\varepsilon}(0) \setminus B_{R-2-\varepsilon}(0)$, respectively.

First we show that the intersections of the boundary circles in ∂S follow the order of the corresponding boundary points:

Lemma 5.1. *Fix $R > 2$. If $z \in \mathcal{O}$ and $d_{\mathrm{H}}(\partial S(z), B_R(0)) \leq \varepsilon$, then as $\varepsilon \downarrow 0$:*

(a) *$z_i \in A_{R-2,\varepsilon}$ for all $1 \leq i \leq n$.*

(b) *The distance between any two points $x, x' \in A_{R-2,\varepsilon}$ such that $\partial B_2(x) \cap \partial B_2(x') \cap A_{R,\varepsilon} \neq \emptyset$ satisfies*

$$|x - x'| \leq 4\sqrt{2\,\frac{R-2}{R}}\,\varepsilon^{1/2} - \frac{\sqrt{2(R-2)}(R-4)}{R^{3/2}}\,\varepsilon^{3/2} + O(\varepsilon^{5/2}), \tag{5.1}$$

and the angle $\theta_{xx'}$ between the rays ℓ_x and $\ell_{x'}$ satisfies

$$|\theta_{xx'}| \leq \frac{4\sqrt{2}}{\sqrt{R(R-2)}}\,\varepsilon^{1/2} + \frac{\sqrt{2}\big(3R(R-2)+8\big)}{3(R(R-2))^{3/2}}\,\varepsilon^{3/2} + O(\varepsilon^{5/2}). \tag{5.2}$$

(c) *For every $1 \leq i \leq n$ there exists a unique $v_i \in \partial B_2(z_i) \cap \partial B_2(z_{i+1})$ such that $v_i \in A_{R.\varepsilon}$.*

(d) *The boundary $\partial S(z)$ consists of the union of closed arcs of the circles $\partial B_2(z_i)$ between the points v_i and v_{i+1}, $1 \leq i \leq n$, contained in $A_{R,\varepsilon}$ (with $v_{n+1} = v_1$).*

Proof. The proof is based on a number of geometric observations.

(a) This claim is immediate from the fact that $\mathrm{dist}_{\mathrm{H}}(\partial S(z), B_R(0)) \leq \varepsilon$ and that each z_i is a boundary point with $B_2(z_i) \subset B_{R+\varepsilon}(0)$ and $\partial B_2(z_i) \cap A_{R,\varepsilon} \neq \emptyset$.

(b) Let $v \in \partial B_2(x) \cap \partial B_2(x') \cap A_{R,\varepsilon}$. To get the bound in (5.1), we note that the maximal distance between x and x' and the maximal angle θ consistent with the condition $v \in A_{R,\varepsilon}$ and $x, x' \in A_{R-2,\varepsilon}$ occur when $v \in \partial B_{R-\varepsilon}(0)$ and $x, x' \in \partial B_{R-2+\varepsilon}(0)$, i.e., when the ray $0\,v$ is orthogonal to the segment $z\,z'$. Assume, without loss of generality, that $v = (R-\varepsilon, 0)$ and $x, x' = (x_0, \pm y_0)$ with $x_0^2 + y_0^2 = (R-2+\varepsilon)^2$. Then the condition $v \in \partial B_2(x) \cap \partial B_2(x')$ reads

$$(R - \varepsilon - x_0)^2 + y_0^2 = 4, \tag{5.3}$$

which, in combination with the equation $x_0^2 + y_0^2 = (R-2+\varepsilon)^2$, yields $x_0 = \frac{R(R-2)-2\varepsilon+\varepsilon^2}{R-\varepsilon}$ and implies

$$\max|x - x'|^2 = 4y_0^2 = \frac{32(R-2)}{R}\varepsilon - \frac{16(R-2)(R-4)}{R^2}\varepsilon^2 + O(\varepsilon^3), \tag{5.4}$$

which settles (5.1). For the corresponding angle, we have $|\tan\frac{\theta}{2}| \leq \frac{y_0}{x_0}$, and hence $|\theta| \leq 2\arctan(\frac{y_0}{x_0})$, which settles (5.2).

(c) Let $\{v_i, v_i'\} = \partial B_2(z_i) \cap \partial B_2(z_{i+1})$ and consider the boundary piece $\partial(B_2(z_i) \cup B_2(z_{i+1}))$. This consists of two arcs, $C_i \subset \partial B_2(z_i))$ and $C_{i+1} \subset \partial B_2(z_{i+1}))$, both ending in the points v_i and v_i'. A necessary condition for both z_i and z_{i+1} to be boundary points is that both arcs C_i and C_{i+1} intersect the annulus $A_{R,\varepsilon}$. If, in addition, $v_i, v_i' \notin A_{R,\varepsilon}$, then we get a contradiction with the assumption that both z_i and z_{i+1} are boundary points. Indeed, consider the line ℓ through v_i' and v_i (and through $\overline{z_i} = \frac{1}{2}(z_i + z_{i+1})$) and the intersection point $p_1 = \ell \cap \partial B_{R-\varepsilon}(0)$ as shown in Fig. 6.

There exists $j \neq i, i+1$ such that $p_1 \in B_2(z_j)$. Otherwise there would be a gap in the boundary $\partial S(z)$ along $\ell \cap A_{R,\varepsilon}$. Assuming, without loss of generality, that the line segment $z_i\,z_j$ intersects the ray ℓ_{i+1} (in view of (5.1), this segment cannot intersect both ℓ_{i+1} and ℓ_{i-1}), we conclude that if $p_1 \in B_2(z_j)$, then also $p_2 \in B_2(z_j)$, where p_2 is the reflection of p_1 with respect to the ray ℓ_{i+1}. But

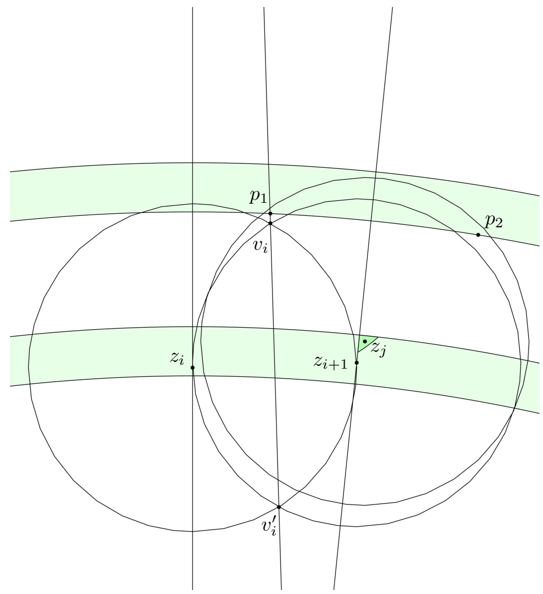

Figure 6: The boundary points z_i and z_{i+1} and the intersection v_i of their halo circles. The upper circle on the right has centre z_j. The dark shaded region is the intersection of the disc $B_2(p_1)$, the annulus $A_{R-2,\varepsilon}$ and the halfplane to the right of ℓ_{i+1}. The light shaded regions are the annuli $A_{R,\varepsilon}$ and $A_{R-2,\varepsilon}$.

then also $\partial B_2(z_{i+1}) \cap A_{R,\varepsilon} \subset B_2(z_j)$, which is in contradiction with the fact that z_{i+1} is a boundary point. Note that there is a severe restriction on the position of the point z_j: it has to be contained in $B_2(p_1)$. The allowed region is shown in Fig. 6 in a darker shade. Thus, necessarily, $v_i \in A_{R,\varepsilon}$, while $v'_i \in B_{R-2-\varepsilon}$, because v'_i is a reflection of v_i with respect to $\overline{z_i} \in A_{R-2,\varepsilon}$.

(d) If the arc of the circle $\partial B_2(z_i)$ between the points v_i and v_{i+1} is intersected by a circle $\partial B_2(z_j)$ for some $j \notin \{z_{i-1}, z_i, z_{i+1}\}$, then necessarily $\{v_i, v_{i+1}\} \cap B_2(z_j) \neq \emptyset$. Similarly as above, assuming that the line segment $z_i\, z_j$ intersects the ray ℓ_{i+1} and knowing that $v_i \in A_{R,\varepsilon} \cap B_2(z_j)$, we get that also its reflection with respect to the ray ℓ_{i+1} belongs to $B_2(z_j)$, which implies that $(\partial B_2(z_{i+1}) \backslash B_2(z_i)) \cap A_{R,\varepsilon} \subset B_2(z_j)$, in contradiction with the fact that z_{i+1} is a boundary point. $\square$

Let

$$\rho_i = r_i - (R-2), \tag{5.5}$$

and note that $r_{i+1} - r_i = \rho_{i+1} - \rho_i$. Abbreviate

$$\begin{aligned} &\overline{z_i} = \tfrac{1}{2}(z_i + z_{i+1}), \quad \overline{\rho_i} = \tfrac{1}{2}(\rho_i + \rho_{i+1}), \\ &\overline{\rho_i \cos t_i} = \tfrac{1}{2}(\rho_i \cos t_i + \rho_{i+1} \cos t_{i+1}), \quad \overline{\rho_i \sin t_i} = \tfrac{1}{2}(\rho_i \sin t_i + \rho_{i+1} \sin t_{i+1}). \end{aligned} \tag{5.6}$$

Proposition 5.2 (A priori estimates for angular and radial coordinates)**.**
Fix $R > 2$. If $z \in \mathcal{O}$ and $d_{\mathrm{H}}(\partial S(z), \partial B_{R_c}(0)) \leq \varepsilon$, then as $\varepsilon \downarrow 0$,

$$\max_{1\leq i\leq n} |\rho_i| = O(\varepsilon), \quad \max_{1\leq i\leq n} \theta_i = O(\sqrt{\varepsilon}\,), \quad \max_{1\leq i\leq n} |\rho_{i+1} - \rho_i|/\theta_i = O(\sqrt{\varepsilon}\,), \quad n^{-1} = O(\sqrt{\varepsilon}). \tag{5.7}$$

Proof. The first estimate in (5.7) is a trivial consequence of $d_{\mathrm{H}}(\partial S(z), \partial B_R(0)) \leq \varepsilon$. The second estimate is the bound (5.2) in Lemma 5.1(b). The fourth estimate is a consequence of the second estimate. Indeed, if $\max_{1\leq i\leq n} \theta_i = O(\sqrt{\varepsilon}\,)$, then $n^{-1} \leq (2\pi)^{-1} \max_{1\leq i\leq n} \theta_i = O(\sqrt{\varepsilon}\,)$.

The third estimate is slightly more involved. Omit the index i, similarly as in the proof of Lemma 5.1(c), and consider points $z, z' \in A_{R-2,\varepsilon}$ with polar coordinates $z = (r, 0)$ and $z' = (r', \theta)$, with $\theta > 0$, $\theta = O(\sqrt{\varepsilon}\,)$, $r' = r - \delta$, $\delta > 0$. Our aim is to evaluate the fraction $\frac{\delta}{\theta}$.

To maximise δ for fixed θ, we assume that $r = R-2+\varepsilon$ and that the point z' is chosen so that a point v on an intersection $\partial B_2(z) \cap \partial B_2(z')$ lies on the inner boundary of $A_{R,\varepsilon}$, $|v| = R-\varepsilon$. We need to find $v = (x,y) \in \partial B_{R+\varepsilon}(0) \cap \partial B_2(z)$ satisfying the equations

$$x^2 + y^2 = (R-\varepsilon)^2, \qquad (x-(R-2+\varepsilon))^2 + y^2 = 4, \tag{5.8}$$

which yield

$$x = \frac{R(R-2) - 2\varepsilon + \varepsilon^2}{R-2+\varepsilon}, \qquad y = \sqrt{(R-\varepsilon)^2 - \Big(\frac{R(R-2)-2\varepsilon+\varepsilon^2}{R-2+\varepsilon}\Big)^2} = 2\frac{\sqrt{R(R-2)(2-\varepsilon)\varepsilon}}{R-2+\varepsilon}. \tag{5.9}$$

The fact that $v \in \partial B_2(z')$ implies that δ determining the position of the point z' satisfies the equation

$$(x-(R-2+\varepsilon-\delta)\sin\theta)^2 + (y-(R-2+\varepsilon-\delta)\cos\theta))^2 = 4 \tag{5.10}$$

Solving for $\delta = \delta(R,\varepsilon,\theta)$ and expanding into powers in ε and θ, we obtain

$$\begin{aligned} \delta/\theta &= -\frac{R(R-2)}{4}\,\theta + \sqrt{2}\sqrt{R(R-2)}\,\sqrt{\varepsilon} - \frac{3R^2+4R-2}{4}\,\theta\varepsilon + O(\sqrt{\varepsilon}\,\theta^2) + O(\theta^3) \\ &\le \sqrt{2R(R-2)}\,\sqrt{\varepsilon} + O(\sqrt{\varepsilon}\,\theta^2) + O(\theta^3), \end{aligned} \tag{5.11}$$

where we drop two negative terms. □

In the sequel we will need four sums involving θ_i, ρ_i, which we collect here.

Definition 5.3. *Fix $R > 2$. Recall (4.4) and (5.5)–(5.6). Define*

$$\begin{aligned} y_1(z) &= \sum_{i=1}^n \theta_i^3, \qquad y_2(z) = \sum_{i=1}^n (\rho_{i+1}-\rho_i)^2/\theta_i, \quad y_3(z) = \sum_{i=1}^n \overline{\rho_i}^2\theta_i, \\ y_4(z) &= \sum_{i=1}^n \overline{\rho_i}\,\theta_i, \quad y_5(z) = \sum_{i=1}^n \theta_i\,\overline{\rho_i\cos t_i}, \qquad y_6(z) = \sum_{i=1}^n \theta_i\,\overline{\rho_i\sin t_i}. \end{aligned} \tag{5.12}$$

These expressions will appear in the expansions in Propositions 5.8 and 5.9 below. The following estimates will play a crucial role in the sequel. Note that $y_1(z), y_2(z), y_3(z)$ are non-negative, while $y_4(z), y_5(z), y_6(z)$ are not necessarily so.

Corollary 5.4 (A priori estimates for sums in approximations)**.**
Fix $R > 2$. If $z \in \mathcal{O}$ and $d_{\mathrm{H}}(\partial S(z), \partial B_{R_c}(0)) \le \varepsilon$, then as $\varepsilon \downarrow 0$,

$$y_\ell(z) = O(\varepsilon), \quad \ell = 1,2,4,5,6, \qquad y_3(z) = O(\varepsilon^2). \tag{5.13}$$

Proof. Using the bounds (5.7), we estimate

$$\begin{aligned} &0 \le y_1(z) \le \Big(\max_{1\le i\le n}\theta_i\Big)^2 \sum_{i=1}^n \theta_i = O(\varepsilon), \qquad 0 \le y_2(z) \le \Big(\max_{1\le i\le n}\tfrac{|\rho_{i+1}-\rho_i|}{\theta_i}\Big)^2 \sum_{i=1}^n \theta_i = O(\varepsilon), \\ &0 \le y_3(z) \le \Big(\max_{1\le i\le n}|\rho_i|^2\Big)\sum_{i=1}^n \theta_i = O(\varepsilon^2), \quad |y_4(z)|, |y_5(z)|, |y_6(z)| \le \Big(\max_{1\le i\le n}|\rho_i|\Big)\sum_{i=1}^n \theta_i = O(\varepsilon). \end{aligned} \tag{5.14}$$

□

5.2 Locality for boundary determination

In this section we present a crucial property of the boundary points, namely, their location is constrained only by the location of the two neighbouring boundary points (Proposition 5.7 below). This property will be used in the proof of the lower bound in Theorem 2.6 carried out in Section 8.2 (see, in particular, the proof of Lemma 8.3). It also plays an important role in [23].

Definition 5.5.

(a) *Let $z = (z_i)_{1\le i\le n} = ((r_i \cos t_i, r_i \sin t_i))_{1\le i\le n}$ be a sequence of points in $A_{R-2,\varepsilon}$ in polar coordinates, ordered by angle, i.e., $t_1 < \cdots < t_n$ (and with $z_{n+1} = z_n$). Suppose that for each pair z_i, z_{i+1} there exists a $v_i \in \partial B_2(z_i) \cap \partial B_2(z_{i+1}) \cap A_{R,\varepsilon}$. Note that this condition implies that $S(z)$ is well defined ($S(z)$ is obtained by filling the inner part) and that $\partial S(z) \subset A_{R,\epsilon}$. However, the condition does not necessarily imply that $z \in \mathcal{O}$, because possibly only a subset of z contributes to $\partial S(z)$.*

(b) *For any $1 \le i, j \le n$, let $v_{i,j} \in \partial B_2(z_i) \cap \partial B_2(z_j)$ be such that $|v_{i,j}| = \max\{|v|\colon v \in \partial B_2(z_i) \cap \partial B_2(z_j)$ (if there is a tie, then take the intersection with minimal angle in polar coordinates). In polar coordinates, $v_{i,j} = (r_{i,j} \cos t_{i,j}, r_{i,j} \sin t_{i,j})$, $r_{i,j} \in [R-\epsilon, R+\epsilon]$. Then:*

(i) *A point z_i is* extremal *in z if $\partial B_2(z_i) \cap \partial S(z) \neq \emptyset$.*

(ii) *A sequence z is a* set of boundary points*, i.e., $z \in \mathcal{O}$, if each z_i, $1 \le i \le n$, are extremal.*

(iii) *A triplet (z_i, z_j, z_k) with $t_i < t_j < t_k$ is called an* extremal triplet *if $(B_2(z_j) \setminus B_{R-\epsilon}(0)) \setminus (B_2(z_i) \cup B_2(z_k)) \neq \emptyset$.*

We need the following lemma.

Lemma 5.6. *Let $R > 2 + \frac{\epsilon}{1-\epsilon}$. Consider two points $x, x' \in A_{R-2,\varepsilon}$. Set $\{v, v'\} = \partial B_2(x) \cap \partial B_2(x')$ and suppose that $\{v, v'\} \cap A_{R,\varepsilon} \neq \emptyset$. Then the following hold:*

(i) *Exactly one of the vectors $\{v, v'\} = \partial B_2(x) \cap \partial B_2(x')$, say v, is in $A_{R,\varepsilon}$. The other is in the interior of the ball $B_{R-\epsilon}(0)$.*

(ii) *Let $x = (r \cos t, r \sin t)$, $x' = (r' \cos t', r' \sin t')$ and assume that $t < t'$. Let H be the halfplane with the boundary consisting of the line vv' containing the point x'. Then $B_2(x) \cap H \subset B_2(x') \cap H$.*

(iii) A triplet (z_i, z_j, z_k) with $v_{j,k} \in A_{R,\varepsilon}$ is extremal if and only if $t_{i,j} < t_{j,k}$.

Proof.

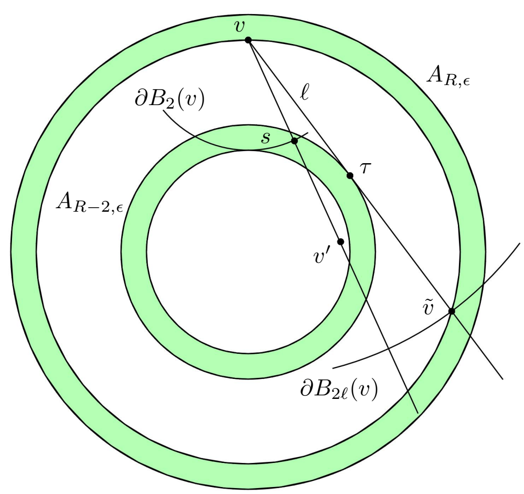

Figure 7: Illustration to the proof that $v' \in B_{R-\epsilon}(0)$.

(i) Choosing a position of v in $A_{R,\varepsilon}$, we see that the points x and x' necessarily lie on the arc $\partial B_2(v) \cap A_{R-2,\varepsilon}$. Thus, the barycentre $s = \frac{1}{2}(x+x')$, which is the centre of symmetry of the union $B_2(x) \cup B_2(x')$, is contained in $\partial B_2(v) \cap A_{R-2,\varepsilon}$. The point v' is symmetric to v with respect to the barycentre s: $v' = s - (v - s)$. Suppose, without loss of generality, that $v = (0, y)$ with $y \in [R-\epsilon, R+\epsilon]$. To show that $v' \in B_{R-\epsilon}(0)$, consider the most extremal case $y = R - \epsilon$ when $|v| = R - \epsilon$. Indeed, for $y > R - \epsilon$

we could shift the points x and x', and thus also v and v', by the vector $u = (0, R-\epsilon-y)$. The shifted $x+u, x'+u$ lead to the shifted $v+u$ and $v'+u$. Notice also that necessarily $x+u, x'+u \in A_{R-2,\epsilon}$ since $x+u, x'+u \in \partial B_2(v) \cap A_{R-2,\epsilon} + u \subset \partial B_2(v+u) \cap A_{R-2,\epsilon}$ in view of the fact that the point $v+u-2 = (0, R-2-\epsilon) \in A_{R-2,\epsilon}$. Now, if $v'+u \in B_{R-\epsilon}(0)$, then necessarily also $v' \in B_{R-\epsilon}(0)$. Consider, in addition, the "most dangerous case" when s is asymptotically approaching the extremal point $B_2(v) \cap \partial B_{R-2+\epsilon}(0)$. Consider the tangent line from v to the disc $B_{R-2+\epsilon}(0)$ touching in the point τ. This tangent line intersects the circle $B_{R-\epsilon}(0)$ in v and a point $\tilde{v}$ symmetric with respect τ. Clearly, if the distance ℓ from v to τ is larger than 2, then the point v' on the line vs falls short of $\partial B_{R-\epsilon}(0)$ and we get the claim. To show that $\ell > 2$, we compute it from the rectangular triangle $v\tau 0$:

$$\ell^2 = (R-\epsilon)^2 - (R-2+\epsilon)^2 = 4(R-1)(1-\epsilon) \tag{5.15}$$

which yields $\ell > 2$ if and only if $R > 2 + \frac{\epsilon}{1-\epsilon}$.

(ii) The claim immediately follows by inspecting the union $B_2(x) \cup B_2(x')$ with the intersection points $\{v, v'\} = \partial B_2(x) \cap \partial B_2(x')$ and the symmetry with respect to the barycentre s.

(iii) Just observe that the condition $t_{i,j} = t_{j,k}$ means that the circle $\partial B_2(x_j)$ is touching the set $\partial B_2(x_i) \cup \partial B_2(x_k)$ in the point $v_{i,k}$. □

Proposition 5.7 (Local characterisation of sets of boundary points)**.**
Let $R > 2 + \frac{\epsilon}{1-\epsilon}$ and $z = ((r_i \cos t_i, r_i \sin t_i))_{1\le i\le n}$ be a sequence of points in $A_{R-2,\varepsilon}$, ordered by angle. Then the following two conditions are equivalent:
(i) The set z is a set of boundary points, $z \in \mathcal{O}$.
(ii) Every triplet (z_{j-1}, z_j, z_{j+1}), $1 \le j \le n$, is extremal.

Proof.

(i) $\implies$ (ii):
If (ii) does not hold, then there exist a j such that the triplet (z_{j-1}, z_j, z_{j+1}) is not extremal. According to Lemma 5.6(iii), this implies that z_j is not extremal in z and the condition (i) is not satisfied.

(ii) $\implies$ (i):
If (i) does not hold, then there exist either $k > j$ or $i < j$ such that either $v_{j-1,j} \in B_2(z_k) \cap A_{R,\varepsilon}$ or $v_{j-1,j} \in B_2(z_i) \cap A_{R,\varepsilon}$. Consider the former case. We will show that, necessarily, one of the triplets

$$(z_{j-1}, z_j z_{j+1}), (z_j, z_{j+1}, z_{j+2}), \dots, (z_{k-2}, z_{k-1}, z_k) \tag{5.16}$$

is not extremal, which breaks (ii). Indeed, if all these triplets were extremal, then we would have $t_{j-1,j} < t_{j,j+1} < t_{j+1,j+2} < \cdots < t_{k-2,k-1} < t_{k-1,k}$. Just observe that $t_{j-1,j} < t_{j,j+1}$ because the triplet (z_{j-1}, z_j, z_{j+1}) is extremal, $t_{j,j+1} < t_{j+1,j+2}$ because the triplet (z_j, z_{j+1}, z_{j+2}) is extremal, etc. Now, given that $t_{k-1} < t_k$ and the fact that the arcs $\partial B_2(z_k) \cap A_{R,\varepsilon}$ and $\partial B_2(z_{k-1}) \cap A_{R,\varepsilon}$ intersect only once at $v_{k-1,k}$, all points $x = (t, \varphi) \in B_2(z_k) \cap A_{R,\varepsilon}$ with $t < t_{k-1,k}$ belong to $B_2(z_{k-1})$. On the other hand, the point $v_{j-1,j} = \partial B_2(z_{j-1}) \cap \partial B_2(z_j) \cap A_{R,\varepsilon}$ does not belong to $B_2(z_{j+1}), B_2(z_{j+2}), \dots, B_2(z_{k-1}), B_2(z_k)$. This is in contradiction with the condition that $v_{j-1,j} \in B_2(z_k)$. □

Proposition 5.7 shows that $1_{\mathcal{O}\cap\mathcal{D}'_\varepsilon(0)}(z)$ is a product of indicators involving triples of successive boundary points. This means that the constraint given by $\mathcal{O} \cap \mathcal{D}'_\varepsilon(0)$ is *nearest-neighbour* only. This simplifying fact will play an important role in the analysis of the effective interface model in [23].

5.3 Volume and surface approximation

In this section we derive approximations of two key quantities:

- $(|S(z)| - \kappa|S^-(z)|) - (\pi R_c^2 - \kappa\pi(R_c-2)^2)$, the volume of the sausage minus the volume of the critical annulus (see Fig. 5).
- $|S(z)| - \pi R_c^2$, the volume of the halo minus the volume of the *critical disc*.

For fixed $\kappa \in (1,\infty)$, we write $R_c = R_c(\kappa)$ and introduce three explicit constants

$$C_1 = \tfrac{1}{48} R_c^2 (R_c - 2) = \frac{\kappa^2}{6(\kappa-1)^3}, \quad C_2 = R_c = \frac{2\kappa}{\kappa - 1}, \quad C_3 = \frac{1}{R_c - 2} = \frac{\kappa - 1}{2}. \tag{5.17}$$

Proposition 5.8 (Volume and surface approximation)**.** *If $z \in \mathcal{O}$ and $d_{\mathrm{H}}(\partial S(z), \partial B_{R_c}(0)) \leq \varepsilon$, then as $\varepsilon \downarrow 0$,*

$$\begin{aligned} |S(z)| - \pi R_c^2 &= -C_1^\varepsilon y_1(z) + C_3^\varepsilon y_2(z) + \tfrac{1}{2} y_3(z) + C_2 y_4(z), \\ \big(|S(z)| - \kappa |S^-(z)|\big) - \big(\pi R_c^2 - \kappa\pi(R_c-2)^2\big) &= C_1^\varepsilon y_1(z) + C_3^\varepsilon y_2(z) - C_3 y_3(z), \end{aligned} \tag{5.18}$$

where $C_k^\varepsilon = [1 + O(\varepsilon)]\, C_k,\ k = 1, 3.$

Proof. The proof comes in 5 steps.

1. The halo $S(z)$ is a disjoint union of n sets of area V_i labelled by the boundary points z_i, $|S(z)| = \sum_{i=1}^n V_i$. Each of these sets is in its turn a disjoint union of 3 subsets: the triangle $0\, z_i\, z_{i+1}$ of $V_i^{(1)}$, the isosceles triangle $z_i\, z_{i+1}\, v_i$ of area $V_i^{(2)}$, and the boundary wedge $z_i\, v_{i-1}\, v_i$ of area $V_i^{(3)}$ (see Fig. 8). The areas are easily expressed in terms of the corresponding vertex angles, namely,

$$V_i = V_i^{(1)} + V_i^{(2)} + V_i^{(3)} = \tfrac{1}{2} r_i r_{i+1} \sin\theta_i + 2\sin\varphi_i + 2\alpha_i, \tag{5.19}$$

with φ_i denoting the angle between the line segments $v_i\, z_i$ and $v_i\, z_{i+1}$ touching at the point v_i, and α_i denoting the angle between the line segments $z_i\, v_{i-1}$ and $z_i\, v_i$ at the point z_i. Similarly, we have $|S^-(z)| = \sum_{i=1}^n V_i^-$, with

$$V_i^- = A_i^{(1)} - B_i = \tfrac{1}{2} r_i r_{i+1} \sin\theta_i - 2(\varphi_i - \sin\varphi_i), \qquad 1 \leq i < n, \tag{5.20}$$

where B_i is the area of the circular segment bounded by the line segment $z_i\, z_{i+1}$ and the arc of the circle $\partial B_2(v_i)$ subtending the angle φ_i.

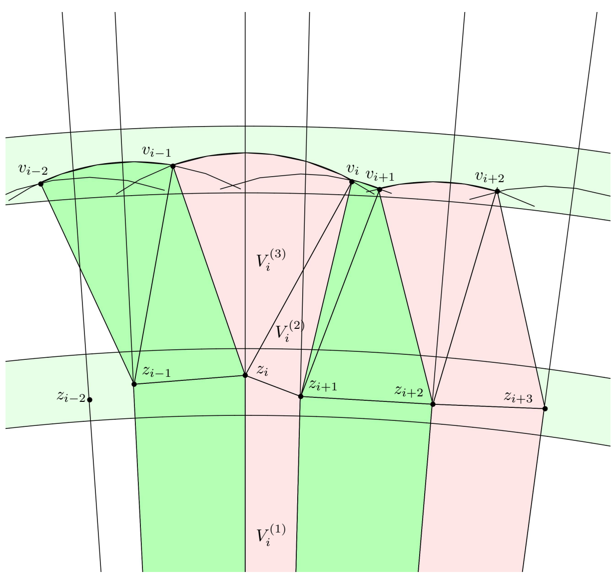

Figure 8: Volumes $V_i^{(1)}$, $V_i^{(2)}$, $V_i^{(3)}$.

It is easy to show that

$$\sum_{i=1}^n \alpha_i = \sum_{i=1}^n (\varphi_i + \theta_i). \tag{5.21}$$

Indeed, let us introduce angles β_i and γ_i at the point z_i between the line ℓ_i and the edges $z_i\, v_{i-1}$ and $z_i\, v_i$, respectively (see Fig. 9). The angle β_i is taken to be positive in the anticlockwise direction from ℓ_i, and γ_i in the clockwise direction. Note that the angle β_{i+1} in Fig. 9 is actually negative, while all the other angles β_j and γ_j are positive. On the one hand, $\beta_i + \gamma_i = \alpha_i$ for $1 \le i \le n$, and in the case of the point z_{i+1} in Fig. 9, $\alpha_{i+1} = \beta_{i+1} + \gamma_{i+1} = \gamma_{i+1} - |\beta_{i+1}|$. Nevertheless, $\alpha_i = \beta_i + \gamma_i > 0$, which is the condition that z_i is a boundary point. On the other hand,

$$\varphi_i + \theta_i = \beta_{i+1} + \gamma_i, \qquad 1 \le i \le n. \tag{5.22}$$

It suffices to consider the triangles $0\, z_i\, z_{i+1}$ with angles are θ_i, $\pi - (\frac{\pi-\varphi_i}{2} + \gamma_i)$ (the complement of the angle $\frac{\pi-\varphi_i}{2} + \gamma_i$ between the edge $z_i\, z_{i+1}$ and the line ℓ_i obtained by adding at z_i the angle γ_i to the angle $\frac{\pi-\varphi_i}{2}$ of the isosceles triangle $z_i\, z_{i+1}\, v_i$), and the angle $\pi - (\frac{\pi-\varphi_i}{2} + \beta_{i+1})$ at z_{i+1}. This yields

$$\theta_i + \pi - (\tfrac{\pi-\varphi_i}{2} + \gamma_i) + \pi - (\tfrac{\pi-\varphi_i}{2} + \beta_{i+1}) = \pi, \tag{5.23}$$

which implies $\varphi_i + \theta_i = \beta_{i+1} + \gamma_i$. Note that this reasoning remains valid for negative β_i or γ_i (check the point z_{i+1} in Fig. 9). Combining (5.22) with the equation $\alpha_i = \beta_i + \gamma_i$, we get

$$\sum_{i=1}^n \alpha_i = \sum_{i=1}^n (\gamma_i + \beta_i) = \sum_{i=1}^n \gamma_i + \sum_{i=1}^n \beta_{i+1} = \sum_{i=1}^n (\varphi_i + \theta_i). \tag{5.24}$$

As a result, we can replace the sum in $|S(z)| = \sum_{i=1}^n V_i$ by the sum $\sum_{i=1}^n V_i'$ with

$$V_i' = \tfrac{1}{2} r_i r_{i+1} \sin\theta_i + 2\sin\varphi_i + 2(\varphi_i + \theta_i). \tag{5.25}$$

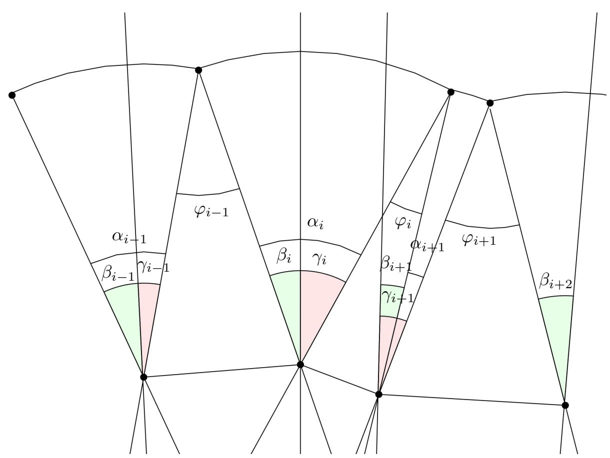

Figure 9: The relations between the angles α_i, β_i, γ_i, θ_i, φ_i.

2. Next, we split the volume of the sausage

$$\big(|S(z)| - \kappa|S^-(z)|\big) - \big(\pi R_c^2 - \kappa\pi(R_c-2)^2\big) = \big[|S(z)| - \pi R_c^2\big] - \kappa\big[|S^-(z)| - \pi(R_c-2)^2\big] \tag{5.26}$$

and rewrite the two terms as

$$\begin{aligned} |S(z)| - \pi R_c^2 &= \sum_{i=1}^n \big(V_i - \tfrac{1}{2} R_c^2 \theta_i\big) = \sum_{i=1}^n (I_i^{(1)} + I_i^{(2)}), \\ |S^-(z)| - \pi(R_c-2)^2 &= \sum_{i=1}^n \big(V_i^- - \tfrac{1}{2}(R_c-2)^2\theta_i\big) = \sum_{i=1}^n (I_i^{(3)} + I_i^{(4)}), \end{aligned} \tag{5.27}$$

where

$$
\begin{aligned}
I_i^{(1)} &= V_i - \tfrac{1}{2}(\overline{r_i}+2)^2\theta_i, & I_i^{(2)} &= \tfrac{1}{2}(\overline{r_i}+2)^2\theta_i - \tfrac{R_c^2}{2}\theta_i,\\
I_i^{(3)} &= V_i^- - \tfrac{1}{2}\overline{r_i}^2\theta_i, & I_i^{(4)} &= \tfrac{1}{2}\overline{r_i}^2\theta_i - \tfrac{1}{2}\textstyle\sum_{i=1}^n \theta_i(R_c-2)^2.
\end{aligned}
\tag{5.28}
$$

Thus, to compute the two quantities in (5.18) we need to compute $\sum_{i=1}^n(I_i^{(1)} - \kappa I_i^{(3)}) + \sum_{i=1}^n(I_i^{(2)} - \kappa I_i^{(4)})$ and $\sum_{i=1}^n I_i^{(1)} + \sum_{i=1}^n I_i^{(2)}$. Our aim now is to expand all the relevant terms in powers of θ_i and $\rho_{i+1} - \rho_i$.

3. For the terms $I_i^{(2)}$ and $I_i^{(4)}$, we use (5.5) to get the identities

$$
I_i^{(2)} = R_c\overline{\rho_i}\theta_i + \tfrac{1}{2}\overline{\rho_i}^2\theta_i, \qquad I_i^{(4)} = (R_c-2)\overline{\rho_i}\,\theta_i + \tfrac{1}{2}\overline{\rho_i}^2\theta_i,
\tag{5.29}
$$

where again $\overline{\rho_i} = \frac{1}{2}(\rho_i + \rho_{i+1})$. Since $R_c = \kappa(R_c - 2)$ and $C_3 = \frac{1}{2}(\kappa - 1)$ (recall (5.17)), this in turn yields

$$
I_i^{(2)} - \kappa I_i^{(4)} = -C_3\overline{\rho_i}^2\theta_i,
\tag{5.30}
$$

which accounts for the third term in the right-hand side of the first line of (5.18). Also, recalling the notation $C_2 = R_c$ (recall (5.17)), we see that the first equality in (5.29) reads

$$
I_i^{(2)} = C_2\overline{\rho_i}\,\theta_i + \tfrac{1}{2}\overline{\rho_i}^2\theta_i,
\tag{5.31}
$$

which accounts for the last two terms in the right-hand side of the second line of (5.18).

4. The terms $I_i^{(1)}$ and $I_i^{(3)}$ are a bit more elaborate and require an expansion of some terms. Begining with $I_i^{(3)}$, we use (5.20) to rewrite the definition (5.28) as

$$
I_i^{(3)} = \tfrac{1}{2}r_ir_{i+1}\sin\theta_i - 2(\varphi_i - \sin\varphi_i) - \tfrac{1}{2}\overline{r_i}^2\theta_i.
\tag{5.32}
$$

Then, using $r_ir_{i+1} = (\overline{r_i} - \frac{1}{2}(\rho_{i+1}-\rho_i))(\overline{r_i} + \frac{1}{2}(\rho_{i+1}-\rho_i)) = \overline{r_i}^2 - (\frac{\rho_{i+1}-\rho_i}{2})^2$, we write the difference between the first and the third term in (5.32) as

$$
\begin{aligned}
\tfrac{1}{2}r_ir_{i+1}\sin\theta_i - \tfrac{1}{2}\overline{r_i}^2\theta_i &= -\tfrac{1}{2}\overline{r_i}^2(\theta_i - \sin\theta_i) - \tfrac{1}{2}\big(\tfrac{\rho_{i+1}-\rho_i}{2}\big)^2\sin\theta_i\\
&= -\tfrac{1}{12}\overline{r_i}^2(\theta_i^3 + O(\theta_i^5)) - \tfrac{1}{2}\big(\tfrac{\rho_{i+1}-\rho_i}{2}\big)^2\sin\theta_i.
\end{aligned}
\tag{5.33}
$$

With the shorthand notation

$$
u_i = |z_i - z_{i+1}|
\tag{5.34}
$$

for the length of the line segment $z_i\,z_{i+1}$, we express this in terms of the angle φ_i in the isosceles triangle $z_i\,z_{i+1}\,v_i$,

$$
u_i = 4\sin(\tfrac{\varphi_i}{2}).
\tag{5.35}
$$

Inverting (5.35), we get

$$
\varphi_i = \tfrac{1}{2}u_i + \tfrac{1}{192}u_i^3 + O(u_i^5)
\tag{5.36}
$$

and thus

$$
2(\varphi_i - \sin\varphi_i) = \tfrac{1}{24}u_i^3 + O(u_i^5),
\tag{5.37}
$$

which together with (5.32) and (5.33) implies

$$
I_i^{(3)} = -\tfrac{1}{12}\overline{r_i}^2\theta_i^3 - \tfrac{1}{24}u_i^3 - \tfrac{1}{2}\big(\tfrac{\rho_{i+1}-\rho_i}{2}\big)^2\sin\theta_i + O(u_i^5) + \overline{r_i}^2O(\theta_i^5).
\tag{5.38}
$$

Furthermore, substituting V_i from (5.25) into $I_i^{(1)}$ defined in (5.28) and comparing it with $I_i^{(3)}$ in (5.32), we get

$$
I_i^{(1)} = I_i^{(3)} + 4\varphi_i - 2\overline{r_i}\theta_i.
\tag{5.39}
$$

By (4.4) and (5.34), we have

$$
u_i = (r_i^2 + r_{i+1}^2 - 2r_ir_{i+1}\cos\theta_i)^{1/2}.
\tag{5.40}
$$

Approximating u_i by $2\overline{r_i}\sin(\frac{\theta_i}{2})$, we express the error E_i as

$$E_i = u_i - 2\overline{r_i}\sin(\tfrac{\theta_i}{2}) = 2\overline{r_i}\sin(\tfrac{\theta_i}{2})\left(\sqrt{1+\frac{(\rho_{i+1}-\rho_i)^2\cos^2(\frac{\theta_i}{2})}{[2\overline{r_i}\sin(\frac{\theta_i}{2})]^2}}-1\right). \tag{5.41}$$

Whenever $z \in \mathcal{O}$ and $d_{\mathrm{H}}(\partial S(z), \partial B_{R_c}(0)) \le \varepsilon$, we can use Proposition 5.2, which implies that $\overline{r_i} = R_c - 2 + O(\varepsilon)$, $\max_{1\le i\le n}|\rho_i| = O(\varepsilon)$, $\max_{1\le i\le n}\theta_i = O(\sqrt{\varepsilon})$, and $\max_{1\le i\le n}|\rho_{i+1}-\rho_i|/\theta_i = O(\sqrt{\varepsilon})$. With help of these bounds, we get

$$\frac{(\rho_{i+1}-\rho_i)^2\cos^2(\frac{\theta_i}{2})}{[2\overline{r_i}\sin(\frac{\theta_i}{2})]^2} = [1+o(1)]\,\frac{(\rho_{i+1}-\rho_i)^2}{\theta_i^2} = O(\varepsilon) \tag{5.42}$$

and thus

$$E_i = 2\overline{r_i}\sin(\tfrac{\theta_i}{2})\frac{1}{2}\frac{(\rho_{i+1}-\rho_i)^2\cos^2(\frac{\theta_i}{2})}{[2\overline{r_i}\sin(\frac{\theta_i}{2})]^2}[1+O(\varepsilon)] = \frac{1}{2}\frac{(\rho_{i+1}-\rho_i)^2}{\overline{r_i}\theta_i}[1+O(\varepsilon)] = O(\varepsilon^{3/2}). \tag{5.43}$$

For u_i as determined by (5.41), where we keep explicitly the terms up to order $O(\varepsilon^{3/2})$, we get

$$u_i = 2\overline{r_i}\sin(\tfrac{\theta_i}{2}) + E_i = \overline{r_i}\Big(\theta_i - [1+O(\varepsilon)]\tfrac{\theta_i^3}{24} + [1+O(\varepsilon)]\tfrac{1}{2}\frac{(\rho_{i+1}-\rho_i)^2}{\overline{r_i}^2\theta_i}\Big) \tag{5.44}$$

and thus also

$$u_i^3 = (\bar{r}_i\theta_i)^3[1+O(\varepsilon)]. \tag{5.45}$$

Using the last two equations jointly with (5.36), we get

$$4\varphi_i - 2\overline{r_i}\theta_i = \Big(\frac{(\rho_{i+1}-\rho_i)^2}{\overline{r_i}\theta_i} + \tfrac{\overline{r_i}}{48}(\overline{r_i}^2-4)\theta_i^3\Big)[1+O(\varepsilon)]. \tag{5.46}$$

Combining this with (5.38) and (5.39), and absorbing the term $\big(\rho_{i+1}-\rho_i\big)^2\theta_i$ into $\frac{(\rho_{i+1}-\rho_i)^2}{\theta_i}O(\varepsilon)$, we get

$$I_i^{(1)} = \Big(\frac{(\rho_{i+1}-\rho_i)^2}{\overline{r_i}\theta_i} - \tfrac{\overline{r_i}}{48}(\overline{r_i}+2)^2\theta_i^3\Big)[1+O(\varepsilon)] \tag{5.47}$$

and

$$I_i^{(1)} - \kappa I_i^{(3)} = \Big(\frac{(\rho_{i+1}-\rho_i)^2}{\overline{r_i}\theta_i} - \tfrac{\overline{r_i}}{48}(\overline{r_i}+2)\big(2+(1-2\kappa)\overline{r_i}\big)\theta_i^3\Big)[1+O(\varepsilon)]. \tag{5.48}$$

5. Using that $R_c = \kappa(R_c-2)$, $\overline{r_i} = R_c - 2 + O(\varepsilon)$, and applying Proposition (5.2) once more while referring to the definitions in (5.17), we arrive at

$$I_i^{(1)} = \left(C_3\frac{(\rho_{i+1}-\rho_i)^2}{\theta_i} - C_1\theta_i^3\right)[1+O(\varepsilon)] \tag{5.49}$$

and

$$I_i^{(1)} - \kappa I_i^{(3)} = \left(C_3\frac{(\rho_{i+1}-\rho_i)^2}{\theta_i} + C_1\theta_i^3\right)[1+O(\varepsilon)]. \tag{5.50}$$

Recalling (5.26)–(5.27), inserting (5.30) and (5.50), and summing over i, we find the first expansion in (5.18). Recalling (5.27), inserting (5.31) and (5.49), and summing over i, we find the second expansion in (5.18). □

5.4 Geometric centre of a droplet

For $z \in \mathcal{O}$, we define the *geometric centre* of the halo shape $S(z)$ as

$$\mathcal{C}(z) = \frac{1}{\sum_{i=1}^n u_i} \sum_{i=1}^n \overline{z_i} u_i, \tag{5.51}$$

where $\overline{z_i} = \frac{1}{2}(z_i + z_{i+1})$. The centre $\mathcal{C}(z)$ may be thought of as the baricentre of the boundary $\partial P(z)$ of the polygon $P(z)$ obtained by connecting the boundary points z_i with the line segment of length u_i connecting z_i and z_{i+1} (see Fig. 7), and $\sum_{i=1}^n u_i$ is the perimeter of $P(z)$. This notion of centre is adopted for mathematical convenience, the principal feature that we need is that to leading order, the centre is given by the discretized Fourier coefficients $\frac{1}{\pi} y_5$ and $\frac{1}{\pi} y_6$. Other definitions of centre might work equally well, but we stock with the above.

We will need the fact that the centre $\mathcal{C}(z)$ is not too far from the origin.

Proposition 5.9 (Centre approximation)**.**
Let $P(z)$ be the polygon associated with z. If $z \in \mathcal{O}$ and $d_{\mathrm{H}}(\partial S(z), \partial B_{R_c}(0)) \leq \varepsilon$, then as $\varepsilon \downarrow 0$,

$$\mathcal{C}(z) = (\Sigma_1(z), \Sigma_2(z)) \tag{5.52}$$

with (recall Definition 5.3)

$$\begin{aligned} \Sigma_1(z) &= \frac{1}{\pi}\, y_5(z) + O(y_1(z)) + O(y_2(z)) + O(y_3(z)) + O(|y_4(z)|), \\ \Sigma_2(z) &= \frac{1}{\pi}\, y_6(z) + O(y_1(z)) + O(y_2(z)) + O(y_3(z)) + O(|y_4(z)|). \end{aligned} \tag{5.53}$$

Proof. We give the proof for Σ_1 only. The proof for Σ_2 is similar. Recall that, in polar coordinates, $z_i = (r_i \cos t_i, r_i \sin t_i)$ and $t_{i+1} - t_i = \theta_i$. We begin by writing

$$\sum_{i=1}^n \theta_i \, \overline{\cos t_i} - \int_0^{2\pi} \mathrm{d}t \, \cos t = \sum_{i=1}^n \int_{t_i}^{t_{i+1}} \mathrm{d}t \left(\overline{\cos t_i} - \cos t\right), \tag{5.54}$$

where in the left-hand side we have simply subtracted 0. Substituting $\tau = t - \bar{t}_i$, we get

$$\begin{aligned} &\int_{t_i}^{t_{i+1}} \mathrm{d}t \left(\overline{\cos t_i} - \cos t\right) = \int_{-\frac{\theta_i}{2}}^{\frac{\theta_i}{2}} \mathrm{d}\tau \left(\tfrac{1}{2}[\cos(\bar{t}_i + \tfrac{\theta_i}{2}) + \cos(\bar{t}_i - \tfrac{\theta_i}{2})] - \cos(\bar{t}_i + \tau)\right) \\ &= \int_{-\frac{\theta_i}{2}}^{\frac{\theta_i}{2}} \mathrm{d}\tau \left(\cos \bar{t}_i \cos \tfrac{\theta_i}{2} - \cos(\bar{t}_i + \tau)\right) = \int_{-\frac{\theta_i}{2}}^{\frac{\theta_i}{2}} \mathrm{d}\tau \left(\cos \bar{t}_i (\cos \tfrac{\theta_i}{2} - \cos \tau) + \sin \bar{t}_i \sin \tau\right) \\ &= \cos \bar{t}_i \int_{-\frac{\theta_i}{2}}^{\frac{\theta_i}{2}} \tfrac{1}{2}\tau^2 \, \mathrm{d}\tau + O(\theta_i^3) = O(\theta_i^3). \end{aligned} \tag{5.55}$$

Note that in passing to the third line we used that $\int_{-\theta_i/2}^{\theta_i/2} \sin \tau \, d\tau = 0$. Consequently,

$$\sum_{i=1}^n \theta_i \, \overline{\cos t_i} = O\left(\sum_{i=1}^n \theta_i^3\right). \tag{5.56}$$

Next, define

$$(\Sigma_1', \Sigma_2') = \sum_{i=1}^n \overline{z_i} u_i. \tag{5.57}$$

Use (5.44) to write

$$\Sigma_1' = \sum_{i=1}^n \overline{r_i \cos t_i}\, \overline{r_i} \left(\theta_i - [1 + O(\varepsilon)] \tfrac{\theta_i^3}{24} + [1 + O(\varepsilon)] \tfrac{1}{2} \frac{(\rho_{i+1} - \rho_i)^2}{\overline{r_i}^2 \theta_i}\right). \tag{5.58}$$

Substituting $r_i = (R_c - 2) + \rho_i$ (recall (5.5)), we can write

$$\overline{r_i \cos t_i}\,\overline{r_i} = (R_c-2)^2\overline{\cos t_i} + (R_c-2)\overline{\rho_i \cos t_i} + (R_c-2)\overline{\rho_i}\,\overline{\cos t_i} + \overline{\rho_i}\,\overline{\rho_i \cos t_i} \tag{5.59}$$

and insert this expansion into (5.58). We estimate each of the four terms in (5.59) separately:

(1) The term with $(R_c-2)^2\overline{\cos t_i}$ can be estimated via (5.56) and gives $O(y_1(z)) + O(y_2(z))$.

(2) The term with $(R_c-2)\overline{\rho_i \cos t_i}$ gives $(R_c-2)y_5(z) + O(\varepsilon)[O(y_1(z)) + O(y_2(z))]$, where we use the a priori estimates in Proposition 5.2, which imply $|\overline{\rho_i \cos t_i}| = O(\varepsilon)$.

(3) The term with $(R_c-2)\overline{\rho_i}\,\overline{\cos t_i}$ gives

$$(R_c-2)y_5(z) + O(y_1(z))^{1/2}O(y_2(z))^{1/2} + O(\varepsilon)[O(y_1(z)) + O(y_2(z))]. \tag{5.60}$$

Indeed, observe that

$$\overline{\rho_i}\,\overline{\cos t_i} = \overline{\rho_i \cos t_i} - \tfrac{1}{4}(\rho_{i+1}-\rho_i)\,(\cos t_{i+1} - \cos t_i) \tag{5.61}$$

and use the same estimate as in (2) plus the bound

$$\begin{aligned} &-\tfrac{1}{4}\sum_{i=1}^n \theta_i(\rho_{i+1}-\rho_i)(\cos t_{i+1} - \cos t_i) \\ &\quad = O\left(\sum_{i=1}^n \theta_i\ |\rho_{i+1}-\rho_i|\right) = O\left(\sum_{i=1}^n \frac{(\rho_{i+1}-\rho_i)^2}{\theta_i}\right)^{1/2} O\left(\sum_{i=1}^n \theta_i^3\right)^{1/2}. \end{aligned} \tag{5.62}$$

(4) For the term with $\overline{\rho_i}\,\overline{\rho_i \cos t_i}$, note that

$$\begin{aligned} \left|\overline{\rho_i \cos t_i}\right| &= \left|\tfrac{1}{2}\Big(\big(\overline{\rho}_i - \tfrac{\rho_{i+1}-\rho_i}{2}\big)\cos t_i + \big(\overline{\rho}_i + \tfrac{\rho_{i+1}-\rho_i}{2}\big)\cos t_{i+1}\Big)\right| \\ &= \left|\overline{\rho}_i\overline{\cos t_i} + \tfrac{1}{4}(\rho_{i+1}-\rho_i)(\cos t_{i+1} - \cos t_i)\right| \le |\overline{\rho}_i| + \tfrac{1}{2}\varepsilon\theta_i, \end{aligned} \tag{5.63}$$

where we use $|\rho_i| \le \varepsilon$ from the a priori estimates. With the bound (5.63), we can check that overall the term with $\overline{\rho_i}\,\overline{\rho_i \cos t_i}$ gives rise to

$$O(y_3(z)) + O(\varepsilon)[O(y_1(z)) + O(y_2(z))]. \tag{5.64}$$

Collect terms to get

$$\Sigma_1' = 2(R_c-2)y_5(z) + O(y_1(z)) + O(y_2(z)) + O(y_3(z)). \tag{5.65}$$

Finally, use (5.44) to write

$$\sum_{i=1}^n u_i = \sum_{i=1}^n \overline{r_i}\left(\theta_i - [1+O(\varepsilon)]\tfrac{\theta_i^3}{24} + [1+O(\varepsilon)]\tfrac{1}{2}\frac{(\rho_{i+1}-\rho_i)^2}{\overline{r_i}^2\theta_i}\right). \tag{5.66}$$

A similar substitution gives

$$\sum_{i=1}^n u_i = 2\pi(R_c-2) + O(y_1(z)) + O(y_2(z)) + O(|y_4(z)|), \tag{5.67}$$

where we used that $\sum_{i=1}^n \theta_i = 2\pi$. Combine (5.51), (5.57), (5.65) and (5.67) to get the claim. □

An immediate consequence of Corollary 5.4 and Propositions 5.8–5.9 is the following.

Corollary 5.10 (A priori estimates volume, surface and centre)**.**
If $z \in \mathcal{O}$ and $d_{\mathrm{H}}(\partial S(z), \partial B_{R_c}(0)) \le \varepsilon$, then as $\varepsilon \downarrow 0$,

$$|S(z)| - \pi R_c^2 = O(\varepsilon), \quad \big(|S(z)| - \kappa|S^-(z)|\big) - \big(\pi R_c^2 - \kappa\pi(R_c-2)^2\big) = O(\varepsilon), \tag{5.68}$$

and

$$\|\mathcal{C}(z)\|_\infty = \max\{|\Sigma_1|, |\Sigma_2|\} = O(\varepsilon). \tag{5.69}$$

6 Stochastic geometry: representation of probabilities as surface integrals

In Section 6.1 we prove that the contribution to the free energy coming from halos that are *not* close to a critical disc either in volume or in Hausdorff distance is negligible (Lemma 6.1), and that the centre of the critical disc can be placed at the origin (Lemma 6.2). In Section 6.2 we show how to rewrite the integral over halo shapes in (1.14) representing the free energy of the critical droplet, by tracking the *boundary points* (Lemma 6.3 and Corollary 6.4). In Section 6.3 we introduce *auxiliary random variables*, list some of their properties (Lemma 6.5), and rewrite the integral over halo shapes as an expectation of a certain exponential functional over these auxiliary random variables (Proposition 6.6). The latter will serve as the staring point for the analysis in Sections 7–8.

6.1 Only disc-shaped droplets matter

For $\delta, \varepsilon > 0$, define the events

$$\begin{aligned} \mathcal{V}_\delta &= \big\{\gamma \in \Gamma\colon\ |V(\gamma) - \pi R_c^2| \leq \delta\big\},\\ \mathcal{D}_\varepsilon &= \big\{\gamma \in \Gamma\colon\ d_{\mathrm{H}}(\partial h(\gamma), \partial B_{R_c}(x)) < \varepsilon \text{ for some } x \in \mathbb{T}\big\}, \end{aligned} \tag{6.1}$$

i.e., the events where the halo is δ-close to a critical disc in volume and ε-close to a critical disc in Hausdorff distance. From Theorem 2.2 we know that, for δ, ε small enough, on the event $\mathcal{V}_\delta \cap \mathcal{D}_\varepsilon$, $h(\gamma)^-$ is connected and simply connected. First we check that we need not worry about $\mathcal{V}_\delta \cap \mathcal{D}_\varepsilon^c$ with $\mathcal{D}_\varepsilon^c = \Gamma \setminus \mathcal{D}_\varepsilon$. Put

$$\delta(\beta) = \beta^{-2/3}. \tag{6.2}$$

Lemma 6.1. *For every $C \in (0,\infty)$ and all $\varepsilon > 0$ small enough, there exists a $\eta(\varepsilon) > 0$ (independent of C) such that, as $\beta \to \infty$,*

$$\mu_\beta\Big(|V(\gamma) - \pi R_c^2| \leq C\beta^{-2/3},\, \gamma \notin \mathcal{D}_\varepsilon\Big) \leq \mathrm{e}^{-\beta[(1+o(1)][I^*(\pi R_c^2)+\eta(\varepsilon)]}. \tag{6.3}$$

Proof. Fix $C \in (0,\infty)$ and $\varepsilon > 0$. Note that $\mathcal{V}_{C\beta^{-2/3}} \subset \mathcal{V}_\varepsilon$ for β large enough. Since $\mathcal{V}_\varepsilon \cap \mathcal{D}_\varepsilon^c$ is closed, we can use the large deviation principle for the halo shape and the halo volume, derived in Theorems 2.1 and 2.3, to bound

$$\limsup_{\beta\to\infty} \frac{1}{\beta} \log \mu_\beta\Big(|V(\gamma) - \pi R_c^2| \leq C\beta^{-2/3},\, \gamma \notin \mathcal{D}_\varepsilon\Big) \leq -\inf_{S \in \mathcal{V}_\varepsilon \cap \mathcal{D}_\varepsilon^c} I(S). \tag{6.4}$$

In view of (2.3) and (2.4), we are left with the minimisation problem

$$\inf \Big\{|S| - \kappa|S^-|\colon\ S \in \mathcal{V}_\varepsilon \cap \mathcal{D}_\varepsilon^c\Big\}. \tag{6.5}$$

If $|S| = \pi R^2$, then on the event $\mathcal{V}_\varepsilon$ we have $|R^2 - R_c^2| \leq \frac{\varepsilon}{\pi}$. Let η be such that $\varepsilon = \sqrt{4R\eta + \eta^2}$. Then, by (2.7)–(2.8) in Theorem 2.2, on the event $\mathcal{D}_\varepsilon^c$ we have $|S| - \kappa|S^-| \geq (\pi R^2 - \kappa\pi(R-2)^2) + 2\pi\kappa\eta$. But $(\pi R^2 - \kappa\pi(R-2)^2) - (\pi R_c^2 - \kappa\pi(R_c-2)^2) = -\pi(\kappa-1)(R-R_c)^2$, while $\eta = \frac{\varepsilon^2}{4R_c} + O(\varepsilon^3)$ and $(R-R_c)^2 = [\frac{\varepsilon}{2\pi R_c} + O(\varepsilon^2)]^2$ for $\varepsilon \downarrow 0$. Therefore we get

$$|S| - \kappa|S^-| \geq (\pi R_c^2 - \kappa\pi(R_c-2)^2) + \eta(\varepsilon), \qquad \eta(\varepsilon) = \frac{\pi}{4}(\kappa-1)\,\varepsilon^2\Big[1 - \frac{1}{4\pi^2}\Big(\frac{\kappa-1}{\kappa}\Big)^2\Big] + O(\varepsilon^3), \tag{6.6}$$

where we use that $R_c = 2\kappa/(\kappa-1)$. Consequently, (6.4) yields

$$\limsup_{\beta\to\infty} \frac{1}{\beta} \log \mu_\beta\Big(|V(\gamma) - \pi R_c^2| \leq C\beta^{-2/3},\, \gamma \notin \mathcal{D}_\varepsilon\Big) \leq -I^*(\pi R_c^2) - \eta(\varepsilon) \tag{6.7}$$

with $\eta(\varepsilon) > 0$ for ε small enough. $\square$

Next we check that on the event $\gamma \in \mathcal{D}_\varepsilon = \bigcup_{x\in\mathbb{T}} \mathcal{D}_\varepsilon(x)$, we need only consider only consider droplets that are close to $B_{R_c}(0)$. Here we exploit the translation invariance of the system. For $\delta, \varepsilon > 0$ and $x \in \mathbb{T}$, define the event

$$\mathcal{C}_\delta(x) = \Big\{\gamma \in \Gamma\colon\ \max(\Sigma_1(z), \Sigma_2(z)) \le \delta\Big\}, \tag{6.8}$$

i.e., the centre $\mathcal{C}(z) = (\Sigma_1(z), \Sigma_2(z))$ of the halo shape $S(z)$ is δ-close to x (see the beginning of Section 5.4).

Lemma 6.2. *For every $C \in (0,\infty)$, every $\varepsilon > 0$, suitable $k = k(\varepsilon, C) > 0$ and $\varepsilon' = \varepsilon'(\varepsilon, C)$, and all β sufficiently large,*

$$\mu_\beta\Big(\mathcal{V}_{C\beta^{-2/3}} \cap \mathcal{D}_\varepsilon\Big) \le \mathrm{e}^{o(\beta^{1/3})} \mu_\beta\Big(\mathcal{V}_{C\beta^{-2/3}} \cap \mathcal{D}_{\varepsilon'}(0) \cap \mathcal{C}_{k\beta^{-2/3}}(0)\Big). \tag{6.9}$$

Proof. If $\partial h(\gamma)$ is ε-close in Hausdorff distance to the boundary of a disc of radius R_c, then by Corollary 5.4 the centre $\mathcal{C}(z)$ of $h(\gamma)$ is $(C'\varepsilon)$-close to the centre of that disc for some $C' \in (0,\infty)$. Let $G_\delta \subset \mathbb{T}$ be the grid of linear spacing δ. It follows that

$$\mathbf{1}_{\mathcal{D}_\varepsilon(0)} \le \mathbf{1}_{\mathcal{D}_\varepsilon} \le \sum_{x\in G_\delta} \mathbf{1}_{\mathcal{C}_{2\delta}(x) \cap \mathcal{D}_{(1+C')\varepsilon+\delta}(x)}. \tag{6.10}$$

Because the torus is periodic, every indicator contributes the same. By picking $\delta = k\beta^{-2/3}$ with $k = C/2$ and $\varepsilon' = (1+C')\varepsilon + C\delta(\beta)/2$ we deduce from (6.10) that

$$\mu_\beta\Big(\mathcal{V}_{C\beta^{-2/3}} \cap \mathcal{D}_\varepsilon\Big) \le |G_{k\beta^{-2/3}}|\, \mu_\beta\Big(\mathcal{V}_{C\beta^{-2/3}} \cap \mathcal{D}_{\varepsilon'}(0) \cap \mathcal{C}_{k\beta^{-2/3}}(0)\Big). \tag{6.11}$$

Clearly, $|G_{k\beta^{-2/3}}| = O(\beta^{4/3}) \le \exp(o(\beta^{1/3}))$, hence the proof is complete. □

Lemmas 6.1 and 6.2 leave us with the task of bounding $\mu_\beta(\mathcal{V}_{C\beta^{-2/3}} \cap \mathcal{D}_{\varepsilon'}(0) \cap \mathcal{C}_{k\beta^{-2/3}}(0))$ from above. For the lower bound, it will be enough to bound $\mu_\beta(\mathcal{V}_{C\beta^{-2/3}} \cap \mathcal{D}_\varepsilon(0))$ from below.

6.2 Integration over halo shapes

Lemma 6.3. *Let Π_α be a Poisson point process of intensity α in $\mathbb{T}$. There exists a constant $c > 0$ such that, for $\varepsilon > 0$ small enough and every bounded test function $f\colon \mathcal{S} \to \mathbb{R}$, as $\alpha \to \infty$,*

$$\mathbb{E}\Big[f\big(h(\Pi_\alpha)\big)\, \mathbf{1}_{\{\Pi_\alpha \in \mathcal{D}_\varepsilon(0)\}}\Big] = \Big[1 - O(\mathrm{e}^{-c\alpha})\Big]\, \mathrm{e}^{-\alpha|\mathbb{T}|} \sum_{n\in\mathbb{N}_0} \frac{\alpha^n}{n!} \int_{\mathbb{T}^n} \mathrm{d}z\, f(S(z))\, \mathrm{e}^{\alpha|S(z)^-|} \mathbf{1}_{\mathcal{D}'_\varepsilon(0)}(z). \tag{6.12}$$

Proof. For $z \in \mathcal{O}$, define

$$r_\alpha(S(z)) = \sum_{m\in\mathbb{N}_0} \frac{\alpha^m}{m!} \int_{\mathbb{T}^m} \mathrm{d}y\ \mathbf{1}_{\{h(y\cup z) = S(z)\}}. \tag{6.13}$$

Clearly,

$$\begin{aligned} r_\alpha(S(z)) &= \sum_{m\in\mathbb{N}_0} \frac{\alpha^m}{m!} \int_{\mathbb{T}^m} \mathrm{d}y\ \mathbf{1}_{\{h(y\cup z)\subset S(z)\}} - \sum_{m\in\mathbb{N}_0} \frac{\alpha^m}{m!} \int_{\mathbb{T}^m} \mathrm{d}y\ \mathbf{1}_{\{h(y\cup z)\subsetneq S(z)\}} \\ &= \mathrm{e}^{\alpha|S(z)^-|}\, [1 - p_\alpha(z)], \end{aligned} \tag{6.14}$$

where

$$p_\alpha(z) = \mathbb{P}\Big(h(\Pi_\alpha \cup z) \subsetneq S(z)\ \Big|\ \Pi_\alpha \subset S(z)^-\Big), \tag{6.15}$$

is the probability that the halo has a *hole*. We argue as follows. Any configuration γ with halo $h(\gamma) = S(z)$ must be of the form

$$\gamma = z \,\dot\cup\, y \quad \text{for some} \quad z = \{z_1, \dots, z_n\}, \quad y = \{y_1, \dots, y_{\ell-n}\}, \quad \ell \ge n, \tag{6.16}$$

where y represents the set of *interior points* of the configuration, i.e., $B_2(y_i)\cap\partial S(z)=\emptyset$ for $1\leq i\leq \ell-n$. Therefore, for every bounded test function f,

$$\begin{aligned}
&\mathbb{E}\Big[f\big(h(\Pi_\alpha)\big)\,\mathbf{1}_{\{\Pi_\alpha\in\mathcal{D}_\varepsilon(0)\}}\Big]\\
&\quad= \mathrm{e}^{-\alpha|\mathbb{T}|}\sum_{\ell\in\mathbb{N}_0}\frac{\alpha^\ell}{\ell!}\int_{\mathbb{T}^\ell}\mathrm{d}\gamma\; f(h(\gamma))\,\mathbf{1}_{\mathcal{D}_\varepsilon(0)}(\gamma)\\
&\quad= \mathrm{e}^{-\alpha|\mathbb{T}|}\sum_{\ell\in\mathbb{N}_0}\frac{\alpha^\ell}{\ell!}\sum_{n=0}^{\ell}\binom{\ell}{n}\int_{\mathbb{T}^n}\mathrm{d}z\; f(S(z))\,\mathbf{1}_{\mathcal{D}'_\varepsilon(0)}(z)\int_{\mathbb{T}^{\ell-n}}\mathrm{d}y\;\mathbf{1}_{\{h(y\cup z)=S(z)\}}\\
&\quad= \mathrm{e}^{-\alpha|\mathbb{T}|}\sum_{n\in\mathbb{N}_0}\frac{\alpha^n}{n!}\int_{\mathbb{T}^n}\mathrm{d}z\; f(S(z))\,\mathbf{1}_{\mathcal{D}'_\varepsilon(0)}(z)\sum_{\ell=n}^{\infty}\frac{\alpha^{\ell-n}}{(\ell-n)!}\int_{\mathbb{T}^{\ell-n}}\mathrm{d}y\;\mathbf{1}_{\{h(y\cup z)=S(z)\}}\\
&\quad= \mathrm{e}^{-\alpha|\mathbb{T}|}\sum_{n\in\mathbb{N}_0}\frac{\alpha^n}{n!}\int_{\mathbb{T}^n}\mathrm{d}z\; f(S(z))\,\mathbf{1}_{\mathcal{D}'_\varepsilon(0)}(z)\,r_\alpha(S(z))\\
&\quad= \mathrm{e}^{-\alpha|\mathbb{T}|}\sum_{n\in\mathbb{N}_0}\frac{\alpha^n}{n!}\int_{\mathbb{T}^n}\mathrm{d}z\; f(S(z))\,\mathbf{1}_{\mathcal{D}'_\varepsilon(0)}(z)\,\mathrm{e}^{\alpha|S(z)^-|}\,[1-p_\alpha(z)].
\end{aligned}\tag{6.17}$$

We get the claim in (6.12), provided we show that there exists a $c>0$ such that, for ε small enough and uniformly in $z\in\mathcal{C}'_{C\delta(\beta)}(0)\cap\mathcal{D}'_\varepsilon(0)$,

$$p_\alpha(z)=O(\mathrm{e}^{-c\alpha}).\tag{6.18}$$

The proof of the latter goes as follows. Let $\Pi_\alpha^{S(z)}$ be the Poisson point process on $S(z)^-$ with intensity α. Then

$$\begin{aligned}
p_\alpha(z)&=\mathbb{P}\Big(h\big(\Pi_\alpha^{S(z)}\cup z\big)\subsetneq S(z)\Big)=\mathbb{P}\Big(h\big(\Pi_\alpha^{S(z)}\big)\subsetneq S(z)\setminus h(z)\Big)=\mathbb{P}\Big(h\big(\Pi_\alpha^{S(z)}\big)\subsetneq S(z)^-\Big)\\
&=\mathbb{P}\Big(\exists\, y\in S(z)^-:\ \Pi_\alpha^{S(z)}\cap B_2(y)=\emptyset\Big)\leq\mathbb{P}\Big(\exists\, y\in S(z)^-:\ \Pi_\alpha^{S(z)}\cap[B_2(y)\cap S(z)^-]=\emptyset\Big).
\end{aligned}\tag{6.19}$$

In order to bound the last expression in (6.19), we discretize. As in the proof of Lemma 6.2, we consider the grid $G_\delta\subset\mathbb{T}$ of linear spacing δ. For every $y\in\mathbb{T}$ there exists an $x\in G_\delta$ such that $B_2(y)\supset B_{2-\delta}(x)$. Therefore

$$\mathbb{P}\Big(\exists\, y\in S(z)^-:\ \Pi_\alpha^{S(z)}\cap[B_2(y)\cap S(z)^-]=\emptyset\Big)\leq\sum_{x\in\mathcal{G}_\delta\cap S(z)^-}\mathbb{P}\Big(\Pi_\alpha^{S(z)}\cap[B_{2-\delta}(x)\cap S(z)^-]=\emptyset\Big).\tag{6.20}$$

It is easy to see that there exists a $c=c(R_c)>0$ such that, for ε small enough,

$$z\in\mathcal{D}'_\varepsilon(0)\quad\Longrightarrow\quad\forall\, x\in S(z)^-:\ |B_2(x)\cap S(z)^-|\geq c.\tag{6.21}$$

Analogously, for all $x\in G_\delta\cap S(z)^-$ we have $|B_{2-\delta}(x)\cap S(z)^-|\geq c-O(\delta)$. Combining (6.19)–(6.21), we get

$$p_\alpha(z)\leq|G_\delta|\mathrm{e}^{-c\alpha+O(\delta)}\leq|\mathbb{T}|\delta^{-2}\,\mathrm{e}^{-c\alpha+O(\delta)}.\tag{6.22}$$

Choosing $\delta=c\alpha^{-2/3}$, we get the claim in (6.18). □

We next observe that

$$\mu_\beta\Big(h(\gamma)\in A,\,\gamma\in\mathcal{D}_\varepsilon(0)\Big)=\frac{\mathrm{e}^{\kappa\beta|\mathbb{T}|}}{\Xi_\beta}\,\mathbb{E}\Big[\mathrm{e}^{-\beta V(\Pi_{\kappa\beta})}\,\mathbf{1}_{\{h(\Pi_{\kappa\beta}\in A\}}\,\mathbf{1}_{\{\Pi_{\kappa\beta}\in\mathcal{D}_\varepsilon(0)\}}\Big],\tag{6.23}$$

where we recall that $\Xi_\beta=\mathrm{e}^{-\beta(1-\kappa)|\mathbb{T}|+o(1)}$. Applying Lemma 6.3 with $\alpha=\kappa\beta$ and with test functions of the form

$$f(S)=\mathrm{e}^{-\beta|S|}\mathbf{1}_A(S),\quad A\subset\mathcal{F}\text{ measurable},\tag{6.24}$$

and abbreviating

$$\Delta(z) = \big(|S(z)| - \kappa|S^-(z)|\big) - \big(\pi R_c^2 - \kappa\pi(R_c-2)^2\big), \tag{6.25}$$

we easily deduces (4.3). Specializing the choice of the indicator $\mathbf{1}_A$, we obtain the following corollary. Define the analogue of (6.1) and (6.8) for connected outer contours rather than configurations,

$$\begin{aligned} \mathcal{V}'_\delta &= \big\{z\in\mathcal{O}\colon\ |V(z)-\pi R_c^2|\le\delta\big\},\\ \mathcal{C}'_\delta(0) &= \big\{z\in\mathcal{O}\colon\ \max(\Sigma_1(z),\Sigma_2(z))\le\delta\big\},\\ \mathcal{D}'_\varepsilon(0) &= \big\{z\in\mathcal{O}\colon\ d_{\mathrm{H}}(\partial S(z),\partial B_{R_c}(0))\le\varepsilon\big\}. \end{aligned} \tag{6.26}$$

Corollary 6.4 (Representation as surface integral)**.** *There exists a $c>0$ such that for every $C\in(0,\infty)$ and all $\varepsilon>0$ small enough, as $\beta\to\infty$,*

$$\begin{aligned} \mu_\beta\Big(\mathcal{V}_{C\beta^{-2/3}}\cap\mathcal{D}_\varepsilon(0)\cap\mathcal{C}_{C\beta^{-2/3}}(0)\Big) &= \Big[1-O(\mathrm{e}^{-c\kappa\beta})\Big]\,\mathrm{e}^{-\beta\, I^*(\pi R_c^2)}\,\mathcal{I}^{\mathrm{UB}}(\kappa,\beta;C,\varepsilon),\\ \mu_\beta\Big(\mathcal{V}_{C\beta^{-2/3}}\cap\mathcal{D}_\varepsilon(0)\Big) &= \Big[1-O(\mathrm{e}^{-c\kappa\beta})\Big]\,\mathrm{e}^{-\beta\, I^*(\pi R_c^2)}\,\mathcal{I}^{\mathrm{LB}}(\kappa,\beta;C,\varepsilon) \end{aligned} \tag{6.27}$$

with

$$\begin{aligned} \mathcal{I}^{\mathrm{UB}}(\kappa,\beta;C,\varepsilon) &= \sum_{n\in\mathbb{N}_0}\frac{(\kappa\beta)^n}{n!}\int_{\mathbb{T}^n}\mathrm{d}z\ \mathrm{e}^{-\beta\Delta(z)}\,\mathbf{1}_{\mathcal{V}'_{C\beta^{-2/3}}\cap\mathcal{C}'_{C\beta^{-2/3}}(0)\cap\mathcal{D}'_\varepsilon(0)}(z),\\ \mathcal{I}^{\mathrm{LB}}(\kappa,\beta;C,\varepsilon) &= \sum_{n\in\mathbb{N}_0}\frac{(\kappa\beta)^n}{n!}\int_{\mathbb{T}^n}\mathrm{d}z\ \mathrm{e}^{-\beta\Delta(z)}\,\mathbf{1}_{\mathcal{V}'_{C\beta^{-2/3}}\cap\mathcal{D}'_\varepsilon(0)}(z). \end{aligned} \tag{6.28}$$

6.3 From surface integral to auxiliary random variables

The following notation allows us to avoid indicators that order variables. For $(t,\rho)\in[0,2\pi)^n\times\mathbb{R}^n$ with $t=(t_1,\dots,t_n)$ pairwise distinct, let $\sigma\in\mathfrak{S}_n$ (the set of permutations of $\{1,\dots,n\}$) be such that $0\le t_{\sigma(1)}<\cdots<t_{\sigma(n)}<2\pi$. We abbreviate $(t_{(i)},r_{(i)})=(t_{\sigma(i)},r_{\sigma(i)})$ and $(t_{(n+1)},r_{(n+1)})=(t_{(1)}+2\pi,r_{(1)})$. The reordering depends on the vector t, but for simplicity we suppress the t-dependence from $r_{(i)}$ and $t_{(i)}$.

The following lemma can be viewed as a relation between measures on the space of marked point processes.

Lemma 6.5. *For every non-negative test function f on $([0,\infty)\times[0,2\pi])^n$ with $f(\emptyset)=0$,*

$$\begin{aligned} &\sum_{n\in\mathbb{N}_0}\frac{1}{n!}(\kappa\beta(R_c-2))^n\int_{\mathbb{R}^n}\mathrm{d}r\int_{[0,2\pi)^n}\mathrm{d}t\ \exp\Big(-\beta\frac{\kappa-1}{2}\sum_{i=1}^n\frac{(r_{(i+1)}-r_{(i)})^2}{t_{(i+1)}-t_{(i)}}\Big)\,f\Big(\{(r_i,t_i)\}_{i=1}^n\Big)\\ &\qquad=\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{2\pi G_\kappa\beta^{1/3}}\int_{\mathbb{R}}\mathrm{d}m\ \mathbb{E}\Big[f\Big(\Big\{\Big(\frac{m+B_{T_i}}{\sqrt{(\kappa-1)\beta}},T_i\Big)\Big\}_{i=1}^N\Big)\prod_{i=1}^N\sqrt{2\pi G_\kappa\beta^{1/3}\Theta_i}\Big]. \end{aligned} \tag{6.29}$$

Proof. We revisit the computations from Section 4.4. Put

$$x_i=\sqrt{2(\kappa-1)\beta}\,\rho_i,\qquad \rho_i=r_i-(R_c-2), \tag{6.30}$$

and set $\theta_i=t_{(i+1)}-t_{(i)}$, where $t_{(n+1)}=t_{(1)}+2\pi$, and $\bar f_t(x)=f(\{(r_i,t_i)\}_{i=1}^n)$. Let $P_\theta(x-y)$ be the heat kernel, see (4.19). Proceeding as in Section 4.4, we see that the left-hand side of (6.29) is equal to

$$\sum_{n\in\mathbb{N}_0}\frac{1}{n!}(G_\kappa\beta^{1/3})^n\int_{[0,2\pi)^n}\mathrm{d}t\prod_{i=1}^n\sqrt{(2\pi)\beta^{1/3}G_\kappa\theta_i}\int_{\mathbb{R}^n}\mathrm{d}x\,\bar f_t(x)\prod_{i=1}^n P_{\theta_i}(x_{(i+1)}-x_{(i)}). \tag{6.31}$$

We first evaluate the inner integral for fixed $n\in\mathbb{N}$ and $t\in[0,2\pi)^n$ with $t_1<\cdots<t_n$, so that $(t_{(i)},r_{(i)})=(t_i,r_i)$ and $x_{i+1}=x_i$. Change variables as $x\to(x_1,x'_2,\dots,x'_n)$ with $x'_i=x_i-x_1$. The inner integral becomes

$$\int_{\mathbb{R}^n}\mathrm{d}x\,\bar f_t(x)\prod_{i=1}^n P_{\theta_i}(x_{(i+1)}-x_{(i)})=\int_{\mathbb{R}}\mathrm{d}x_1\int_{\mathbb{R}^{n-1}}\mathrm{d}x'\,\bar f_t(x_1,x_1+x')\prod_{i=1}^n P_{\theta_i}(x'_{i+1}-x'_i), \tag{6.32}$$

where $x'_1 = x'_{n+1} = 0$. From the semi-group property of the heat kernel, we get

$$\int_{\mathbb{R}^{n-1}} \mathrm{d}x' \prod_{i=1}^{n} P_{\theta_i}(x'_{i+1} - x'_i) = P_{\sum_{i=1}^n \theta_i}(x'_{n+1} - x'_1) = P_{2\pi}(0) = \frac{1}{\sqrt{2\pi}}. \tag{6.33}$$

Next we note

$$\int_{\mathbb{R}} \mathrm{d}x_1 \int_{\mathbb{R}^{n-1}} \mathrm{d}x'\, \bar{f}_t(x_1, x_1 + x') \prod_{i=1}^{n} P_{\theta_i}(x'_{i+1} - x'_i) = \sqrt{2\pi}^{-1} \int_{\mathbb{R}} \mathrm{d}x_1\, \mathbb{E}\Big[\bar{f}_t\big(\{x_1 + \widetilde{W}_{t_i}\}_{i=1}^n\big)\Big] \tag{6.34}$$

(think of $x'_i = \widetilde{W}_{t_i}$). Furthermore, for every non-negative test function g on path space,

$$\begin{aligned}\mathbb{E}\Big[\int_{\mathbb{R}} \mathrm{d}x_1\, g(x_1 + \widetilde{W})\Big] &= \mathbb{E}\Bigg[\mathbb{E}\Big[\int_{\mathbb{R}} \mathrm{d}x_1\, g(x_1 + \widetilde{W}) \,\Big|\, (\widetilde{W}_t)_{t\in[0,2\pi]}\Big]\Bigg] \\ &= \mathbb{E}\Bigg[\mathbb{E}\Big[\int_{\mathbb{R}} \mathrm{d}m\, g(m + B) \,\Big|\, (\widetilde{W}_t)_{t\in[0,2\pi]}\Big]\Bigg] = \mathbb{E}\Big[\int_{\mathbb{R}} \mathrm{d}m\, g(m+B)\Big],\end{aligned} \tag{6.35}$$

where we have changed variables $m = x_1 + M$ with $M = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}t\, \widetilde{W}_t$. It follows that

$$\int_{\mathbb{R}^n} \mathrm{d}x\, \bar{f}_t(x) \prod_{i=1}^{n} P_{\theta_i}(x_{(i+1)} - x_{(i)}) = \int_{\mathbb{R}} \mathrm{d}m\, \sqrt{2\pi}^{-1}\, \mathbb{E}\Big[f\Big(\Big\{\Big(\frac{m + B_{t_{(i)}}}{\sqrt{(\kappa-1)\beta}}, t_{(i)}\Big)\Big\}_{i=1}^n\Big)\Big]. \tag{6.36}$$

This holds as well when the t_i's are pairwise distinct but not necessarily labeled in increasing order. The case when $t_i = t_j$ for some $i \neq j$ has Lebesgue measure zero and need not be considered. Denote the value in (6.36) by $g(t)$. Then (6.31) reads

$$\frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{N}_0} \frac{1}{n!}(G_\kappa \beta^{1/3})^n \int_{[0,2\pi)^n} \mathrm{d}t \prod_{i=1}^{n} \sqrt{(2\pi)\beta^{1/3} G_\kappa \theta_i}\; g(t). \tag{6.37}$$

With the help of (2.20)–(2.21), this expression in turn is equal to

$$\frac{1}{\sqrt{2\pi}} \mathrm{e}^{2\pi G_\kappa \beta^{1/3}} \mathbb{E}\Big[g(\{T_i\}_{i=1}^N) \prod_{i=1}^{N} \sqrt{2\pi G_\kappa \beta^{1/3}}\,\Big] \tag{6.38}$$

and the proof is readily concluded. □

The integral over m corresponds to a freedom of choice in the *average height of the surface of the critical droplet* with respect to critical radius R_c, and constitutes a fine tuning of the volume. We will later see that the integral is dominated by values of m that are at most of order $\beta^{1/6}$.

Remember the process $Z^{(m)} = (Z_i^{(m)})_{i=1}^N$ from (2.22) and (2.23) and the random variables $\widehat{Y}_0$ and $\widehat{Y}_1$ from (2.24). Further define (recall the definition of C_1 in (5.17))

$$Y_0 = \widehat{Y}_0, \qquad Y_1 = \sum_{i=1}^{N} \Theta_i^3 = \frac{1}{C_1 \beta}\widehat{Y}_1, \tag{6.39}$$

$$Y_2 = \sum_{i=1}^{N} \frac{(B_{T_{i+1}} - B_{T_i})^2}{\Theta_i}, \quad Y_3^{(m)} = \sum_{i=1}^{N} \big(m + \overline{B_{T_i}}\,\big)^2 \Theta_i. \tag{6.40}$$

Proposition 6.6 (Representation of key surface integrals)**.** *The integrals in* (6.28) *equal*

$$\begin{aligned}\mathcal{I}^{\mathrm{UB}}(\kappa,\beta;C,\varepsilon) &= \mathrm{e}^{2\pi G_\kappa \beta^{1/3} + o(\beta^{1/3})}\, \mathbb{E}[\mathrm{e}^{Y_0 - C_1\beta Y_1}] \\ &\quad\times \int_{\mathbb{R}} \mathrm{d}m\, \widehat{\mathbb{E}}\Big[\mathrm{e}^{O(\varepsilon)(\beta Y_1 + Y_2 + N) + \frac12 Y_3^{(m)}} \mathbf{1}_{\{Z^{(m)} \in \mathcal{V}'_{C\delta(\beta)} \cap \mathcal{C}'_{C\delta(\beta)}(0) \cap \mathcal{D}'_\varepsilon(0)\}}\Big], \\ \mathcal{I}^{\mathrm{LB}}(\kappa,\beta;C,\varepsilon) &= \mathrm{e}^{2\pi G_\kappa \beta^{1/3} + o(\beta^{1/3})}\, \mathbb{E}[\mathrm{e}^{Y_0 - C_1\beta Y_1}] \\ &\quad\times \int_{\mathbb{R}} \mathrm{d}m\, \widehat{\mathbb{E}}\Big[\mathrm{e}^{O(\varepsilon)(\beta Y_1 + Y_2 + N) + \frac12 Y_3^{(m)}} \mathbf{1}_{\{Z^{(m)} \in \mathcal{V}'_{C\delta(\beta)} \cap \mathcal{D}'_\varepsilon(0)\}}\Big].\end{aligned} \tag{6.41}$$

where $C_1 = \frac{\kappa^2}{6(\kappa-1)^3}$.

Proof. Return to (6.28) and recall (5.17), Proposition 5.8 and (4.2)–(6.26). First we rewrite the expression in (6.28) in terms of polar coordinates $z_i = (r_i \cos t_i, r_i \sin t_i)$, $1 \leq i \leq N$, i.e., $\int_{\mathbb{T}^N} \mathrm{d}z$ becomes $\int_{\mathbb{R}^N} \mathrm{d}r \int_{[0,2\pi)^N} \mathrm{d}t \prod_{i=1}^N r_i$. The latter product becomes

$$\prod_{i=1}^N r_i = \prod_{i=1}^N \big[R_c - 2 + \rho_i\big] = [R_c - 2 + O(\varepsilon)]^N = (R_c-2)^n[1+O(\varepsilon)]^N, \tag{6.42}$$

where the last equality uses the constraint $\mathcal{C}'_{\delta(\beta)}(0) \cap \mathcal{D}'_\varepsilon(0)$ together with the a priori estimate from Proposition 5.2. In this way we obtain the factor $(R_c-2)^N$ needed for Lemma 6.5, together with an error term $\exp[O(\varepsilon)N]$, which shows up as the term $O(\varepsilon)N$ in (6.41). The factor $\mathrm{e}^{2\pi G_\kappa \beta^{1/3}}$ is needed to compensate for the exponent in the Poisson distribution of N (recall (2.19)).

The Gaussian density in Lemma 6.5 is obtained from $\exp[-\beta\Delta(z)]$, more precisely, from the second term in the expansion of $\Delta(z)$ in the first line of (5.18), up to an error term $O(\varepsilon)$ in the constant $C_3^\varepsilon = C_3\,[1+O(\varepsilon)]$, which shows up as the term $O(\varepsilon)Y_2$ in (6.41). Hence Lemma 6.5 is applicable. The first and the third term in the expansion of $\Delta(z)$ in the first line of (5.18) give rise to the term $-\beta C_1^\varepsilon Y_1 + \frac{1}{2} Y_3^{(m)}$ (recall (6.30)). The factor $\prod_{i=1}^N \sqrt{2\pi G_\kappa \beta^{1/3}\Theta_i}$ in (6.29) gives rise to Y_0. The indicator is inherited from the original expression for the integral. □

In what follows we abbreviate

$$\Upsilon^{\mathrm{UB}'}_{\beta,\varepsilon} = \mathcal{V}'_{C\delta(\beta)} \cap \mathcal{C}'_{C\delta(\beta)}(0) \cap \mathcal{D}'_\varepsilon(0), \qquad \Upsilon^{\mathrm{LB}'}_{\beta,\varepsilon} = \mathcal{V}'_{C\delta(\beta)} \cap \mathcal{D}'_\varepsilon(0). \tag{6.43}$$

7 Asymptotics of surface integrals: preparations

Our primary task for proving Theorems 2.5–2.6 in Section 8 is the evaluation of the key surface integrals $\mathcal{I}^{\mathrm{UB}}$ and $\mathcal{I}^{\mathrm{LB}}$ in (6.41). In this section we collect some properties of the auxiliary random variables appearing in Section 6.3 that will help us to estimate these integrals. This requires various approximation arguments, including control of *exponential moments* and *discretisation errors*.

In Section 7.1 we look at moderate deviations for the angular process and compute the leading order contribution to the key surface integrals (Proposition 7.1 and Lemma 7.2). In Section 7.2 we analyse the radial process, which is controlled by the mean-centred Brownian bridge introduced in (2.17)–(2.18) (Lemma 7.3), and estimate two exponential moments involving the latter (Lemma 7.4–7.5). In Section 7.3 we focus on discretisation errors that arise because the mean-centred Brownian motion is only observed along the angular process (Lemmas 7.6–7.9).

7.1 Moderate deviations for the angular point process

The key technical result of this section is the following. Remember $\widehat{Y}_0, \widehat{Y}_1$ from (2.24) and Y_0, Y_1 from (6.39), G_κ and $\lambda(\beta) = G_\kappa \beta^{1/3}$ from (2.19) and τ_* from (2.26).

Proposition 7.1 (Leading order prefactor)**.**

$$\lim_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log \mathbb{E}\big[\mathrm{e}^{Y_0 - \beta C_1 Y_1}\big] = -2\pi G_\kappa (1-\tau_*). \tag{7.1}$$

Proof. The proof builds on an underlying renewal structure. Define the probability density

$$q_*(u) = \sqrt{2\pi u}\, \exp\Big(-\tau_* u - \tfrac{1}{24}u^3\Big), \qquad u \in (0,\infty), \tag{7.2}$$

where τ_* is given in (2.26). A close look at the relevant expressions in polar coordinates reveals that

$$\begin{aligned}\mathbb{E}\big[\mathrm{e}^{Y_0-\beta C_1 Y_1}\big] = \mathrm{e}^{-2\pi G_\kappa \beta^{1/3}} \Bigg(1 + \sum_{n\in\mathbb{N}} (G_\kappa \beta^{1/3})^n \frac{2\pi}{n} \int_{[0,2\pi]^{n-1}} \mathrm{d}\theta\, \mathbf{1}_{\{\sum_{i=1}^{n-1}\theta_i \leq 2\pi\}} \\ \times \prod_{i=1}^n \Big(\sqrt{2\pi G_\kappa \beta^{1/3}\theta_i}\ \mathrm{e}^{-\frac{1}{24}G_\kappa \theta_i^3}\Big)\Bigg),\end{aligned} \tag{7.3}$$

where $\theta_n = 2\pi - \sum_{i=1}^{n-1} \theta_i$. (The factor $\frac{2\pi}{n}$ represents the number of ways to rotate a configuration in such a way that the origin falls within in an interval of average length $\frac{2\pi}{n}$, and is similar to a factor appearing in the definition of stationary renewal processes.) Rewrite (7.3) with n-fold convolutions of q_* as

$$\mathbb{E}\big[\mathrm{e}^{Y_0-\beta C_1 Y_1}\big] = \mathrm{e}^{-2\pi G_\kappa \beta^{1/3}} \left(1 + \mathrm{e}^{2\pi G_\kappa \beta^{1/3} \tau_*} G_\kappa \beta^{1/3} \sum_{n\in\mathbb{N}} \frac{2\pi}{n} (q_*)^{*n} (2\pi G_\kappa \beta^{1/3})\right). \tag{7.4}$$

If the factor $\frac{2\pi}{n}$ were absent, then the sum over n would correspond to the probability for a renewal process with interarrival distribution q_* to have a renewal point at time $2\pi G_\kappa \beta^{1/3}$ given that it has a renewal point at time 0. For large time intervals standard renewal theory tells us that the renewal probability converges to the inverse of the expected interarrival time, which is finite. Therefore we may expect the inner sum to converge and the overall expression to behave like a constant times $G_\kappa \beta^{1/3} \exp(-2\pi G_\kappa \beta^{1/3}(1-\tau_*))$. Remember that $\tau_* > 0$, so the contribution from $n = 0$ is negligible.

For the proof of the upper bound in (7.1), we bound the sum over n in (7.4) by

$$2\pi\, \mathcal{R}\big(2\pi G_\kappa \beta^{1/3}\big) \quad \text{with} \quad \mathcal{R}(\ell) = \sum_{n\in\mathbb{N}} (q_*)^{*n}(\ell). \tag{7.5}$$

The quantity $\mathcal{R}(\ell)$ solves the renewal equation

$$\mathcal{R}(\ell) = q_*(\ell) + \int_0^\infty \mathrm{d}y\, q_*(y)\, \mathcal{R}(\ell - y). \tag{7.6}$$

It follows from [16, Theorem 2, Chapter XI.3] and the smoothness of $\ell \mapsto \mathcal{R}(\ell)$ that

$$\lim_{\ell\to\infty} \mathcal{R}(\ell) = \frac{1}{\int_0^\infty \mathrm{d}u\, u\, q_*(u)} \tag{7.7}$$

and hence $\lim_{\ell\to\infty} \frac{1}{\ell} \log \mathcal{R}(\ell) = 0$. Combining this with (7.4) and recalling $\tau_* > 0$, we get

$$\limsup_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log \mathbb{E}\big[\mathrm{e}^{Y_0-\beta C_1 Y_1}\big] \le 2\pi G_\kappa (\tau_* - 1). \tag{7.8}$$

For the proof of the lower bound in (7.1), we drop all except one term from the sum in (7.4), i.e.,

$$\mathbb{E}[\mathrm{e}^{Y_0 - C_1 \beta Y_1}] \ge G_\kappa \beta^{1/3} \mathrm{e}^{-2\pi G_\kappa \beta^{1/3}(1-\tau_*)} \frac{2\pi}{n} (q_*)^{*n} (2\pi G_\kappa \beta^{1/3}). \tag{7.9}$$

This inequality holds for every $n \in \mathbb{N}$, and a proper choice will be made later. Let $(X_i)_{i\in\mathbb{N}}$ be i.i.d. random variables with probability density function q_*. Then $\mathbb{E}[X_1] = \mu_*$ with $\mu_* = \int_0^\infty \mathrm{d}u\, u\, q_*(u)$, and $(\sum_{i=1}^n X_i - n\mu_*)/\sqrt{n}$ has probability density function

$$p_n(y) = \sqrt{n}\, (q_*)^{*n} \big(\sqrt{n}\, [n\mu_* + y]\big). \tag{7.10}$$

Put differently,

$$(q_*)^{*n}(x) = \tfrac{1}{\sqrt{n}}\, p_n\big(\tfrac{x - n\mu_*}{\sqrt{n}}\big). \tag{7.11}$$

By the local central limit theorem for i.i.d. random variables with densities (see [24, Chapter 4.5]), we have

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \left| p_n(y) - \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\tfrac{y^2}{2\sigma^2}\Big) \right| = 0 \tag{7.12}$$

with σ^2 the variance of X_1. We now choose $n = n(\beta) = \lfloor 2\pi G_\kappa \beta^{1/3}/\mu_* \rfloor$. Then $2\pi G_\kappa \beta^{1/3} = n(\beta)\mu_* + o(1)$ and

$$(q_*)^{*n(\beta)}(2\pi G_\kappa \beta^{1/3}) = \frac{1}{\sqrt{n(\beta)}}\, p_{n(\beta)}\big(o(1)\big) = [1 + o(1)]\, \frac{1}{\sqrt{2\pi G_\kappa \beta^{1/3}/\mu_*}} \frac{1}{\sqrt{2\pi\sigma^2}}. \tag{7.13}$$

Consequently,

$$\lim_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log \Big[(q_*)^{*n(\beta)}(2\pi G_\kappa \beta^{1/3})\Big] = 0 \tag{7.14}$$

and so (7.9) gives

$$\liminf_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log \mathbb{E}[\mathrm{e}^{Y_0 - C_1\beta Y_1}] \geq -2\pi G_\kappa(1-\tau_*). \tag{7.15}$$

□

Proposition 7.1 is complemented by the following lemma, which will help us take care of small perturbations.

Lemma 7.2. *As $\delta \downarrow 0$,*

$$\limsup_{\beta\to\infty} \left| \frac{1}{\beta^{1/3}} \log \widehat{\mathbb{E}}\big[\mathrm{e}^{O(\delta)(\beta Y_1 + N)}\big] \right| = O(\delta). \tag{7.16}$$

Proof. Let $c \geq 0$ be an arbitrary constant that does not depend on β. We write $-c\min(1,C_1)\delta \leq O(\delta) \leq c\min(1,C_1)\delta$. Then

$$\log \widehat{\mathbb{E}}\big[\mathrm{e}^{c\delta(\beta C_1 Y_1 + N)}\big] = \log \mathbb{E}\big[\mathrm{e}^{c\delta N + Y_0 - (1-c\delta)\beta C_1 Y_1}\big] - \log \mathbb{E}\big[\mathrm{e}^{Y_0 - \beta C_1 Y_1}\big]. \tag{7.17}$$

The asymptotic behavior of the second term in the difference is given by Proposition 7.1. For the first term, let $\tau_*(c\delta)$ be the solution of

$$\mathrm{e}^{c\delta} \int_0^\infty \sqrt{2\pi u}\, \exp\Big(-\tau_*(c\delta) - \tfrac{1}{24}(1-c\delta)u^3\Big)\, \mathrm{d}u = 1. \tag{7.18}$$

Thus, $\tau_*(0) = \tau_*$. For sufficiently small δ, the solution exists, is unique, and satisfies

$$\tau_*(c\delta) - \tau_* = O(\delta) \quad (\delta \downarrow 0). \tag{7.19}$$

Arguments analogous to the proof of Proposition 7.1 show that

$$\lim_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log \mathbb{E}\big[\mathrm{e}^{c\delta N + Y_0 - (1-c\delta)\beta C_1 Y_1}\big] = -2\pi\big(1 - \tau_*(c\delta)\big). \tag{7.20}$$

Hence (7.17) yields

$$\lim_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log \widehat{\mathbb{E}}\big[\mathrm{e}^{c\delta(\beta C_1 Y_1 + N)}\big] = -2\pi\big(\tau_*(c\delta) - \tau_*\big) = O(\delta). \tag{7.21}$$

A similar argument shows that

$$\lim_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log \widehat{\mathbb{E}}\big[\mathrm{e}^{-c\delta(\beta C_1 Y_1 + N)}\big] = O(\delta). \tag{7.22}$$

□

We close this section with the following observation, which will not be needed in the sequel but is nonetheless instructive. Let $\mathcal{P}(0,\infty)$ denote the space of probability measures on $(0,\infty)$, equipped with the weak topology, and put

$$\mathcal{M} = \left\{ (x,\mu) \in [0,\infty) \times \mathcal{P}(0,\infty) \colon \int_0^\infty \mathrm{d}\mu(\alpha)\, \alpha = 1 \right\}. \tag{7.23}$$

Define

$$L = \frac{1}{N} \sum_{i=1}^N \delta_{N\Theta_i/2\pi}. \tag{7.24}$$

For $(x, \mu) \in \mathcal{M}$ with $x > 0$, define

$$I_{\mathcal{T}}(x, \mu) = (x \log x - x + 1) + xH\big(\mu \mid \mathrm{EXP}(1)\big), \tag{7.25}$$

where EXP(1) is the exponential distribution with parameter 1, and $H(\mu \mid \mathrm{EXP}(1))$ is the relative entropy of μ with respect to EXP(1). For $(0, \mu) \in \mathcal{M}$, define

$$I_{\mathcal{T}}(0, \mu) = \liminf_{\substack{(x,\nu)\to(0,\mu):\\ x>0,\, \nu\in\mathcal{P}(0,\infty)}} I_{\mathcal{T}}(x, \nu), \tag{7.26}$$

Then the family

$$\left(\mathbb{P}\Big(\Big(\frac{N}{2\pi G_\kappa \beta^{1/3}}, L\Big) \in \cdot \mid N \geq 1\Big)\right)_{\beta\geq 1} \tag{7.27}$$

satisfies the weak LDP on $\mathcal{M}$ with rate $2\pi G_\kappa \beta^{1/3}$ and lower semi-continuous rate function $I_{\mathcal{T}}$. (Not all level sets of $I_{\mathcal{T}}$ are compact.)

7.2 Properties of mean-centred Brownian bridge

Covariance of mean-centred Brownian bridge. The following lemma clarifies the nature of the process $(B_t)_{t\in[0,2\pi]}$ and will be used repeatedly later on.

Lemma 7.3.

(a) $(B_t)_{t\in[0,2\pi]}$ *is a Gaussian process with mean* $\mathbb{E}[B_t] = 0$ *and covariance* $\mathbb{E}[B_t B_s] = k(t - s)$*, where*

$$k(t) = \frac{1}{4\pi}\big[(\pi - |t|)^2 - \pi^2\big] + \frac{\pi}{6}. \tag{7.28}$$

(b) *For every continuous function* $f\colon [0, 2\pi] \to \mathbb{R}$*,*

$$\mathbb{E}\big[\mathrm{e}^{\mathrm{i}\int_0^{2\pi} \mathrm{d}t\, f(t) B_t}\big] = \mathrm{e}^{-\frac{1}{2}\langle f, Kf\rangle}, \tag{7.29}$$

with $\langle\cdot,\cdot\rangle$ *the scalar product in* $L^2([0, 2\pi])$*, and* $g(t) = (Kf)(t) = \int_0^{2\pi} \mathrm{d}s\, k(t-s) f(s)$ *the solution of* $-g''(t) = f(t) - \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}s\, f(s)$*,* $g(2\pi) = g(0)$ *and* $\int_0^{2\pi} \mathrm{d}s\, g(s) = 0$*.*

Proof. (a) $(B_t)_{t\in[0,2\pi]}$ is a linear transformation of the Gaussian process $(W_t)_{t\in[0,2\pi]}$ and therefore is itself Gaussian. The mean-zero property of B_t is inherited from W_t. The elementary computation of the covariance is similar to Deheuvels [10, Lemma 2.1]. We provide the details to identify constants. Set $M = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}s\, B_s$. Since $\mathbb{E}[W_t W_s] = \min(s, t)$ we have, for $s \leq t$,

$$\begin{aligned}
\mathbb{E}[\widetilde{W}_s \widetilde{W}_t] &= \min(s, t) - \frac{st}{2\pi} = \frac{1}{2\pi} s(2\pi - t),\\
\mathbb{E}[\widetilde{W}_t M] &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}u\, \Big(\min(u, t) - \frac{ut}{2\pi}\Big) = \frac{1}{4\pi} t(2\pi - t),\\
\mathbb{E}[M^2] &= \frac{1}{8\pi^2}\int_0^{2\pi} \mathrm{d}t\, (2\pi t - t^2) = \frac{\pi}{6},
\end{aligned} \tag{7.30}$$

and hence

$$\begin{aligned}
\mathbb{E}[B_t B_s] &= \min(s, t) - \frac{st}{2\pi} - \frac{1}{2\pi}\Big(-\frac{t^2}{2} + t\pi\Big) - \frac{1}{2\pi}\Big(-\frac{s^2}{2} + s\pi\Big) + \mathbb{E}[M^2]\\
&= \frac{1}{4\pi}\big[\big(\pi - |t - s|\big)^2 - \pi^2\big] + \frac{\pi}{6} = k(t - s).
\end{aligned} \tag{7.31}$$

By symmetry, the identity also holds for $t \leq s$.

(b) (7.29) follows from standard arguments for Gaussian processes. The kernel $k\colon [-2\pi, 2\pi] \to \mathbb{R}$ satisfies

$$k(t) = k(t+2\pi) \quad \forall\, t \in [-2\pi, 0], \qquad \int_0^{2\pi} \mathrm{d}t\, k(t) = 0, \tag{7.32}$$

and is twice differentiable with second derivative $-1/2\pi$, except at $t = 0$ where the first derivative jumps from $+\frac{1}{2}$ to $-\frac{1}{2}$. Let $g = Kf$. Then g has mean zero and satisfies $g(2\pi) = g(0)$. Furthermore,

$$\begin{aligned} g'(t) &= \int_0^t \mathrm{d}s\, k'(t-s) f(s) + \int_t^{2\pi} \mathrm{d}s\, k'(t-s) f(s), \\ g''(t) &= k'(0+) f(t) - k'(0-) f(t) + \tfrac{1}{2\pi} \int_0^{2\pi} \mathrm{d}s\, f(s) = -f(t) + \tfrac{1}{2\pi} \int_0^{2\pi} \mathrm{d}s\, f(s). \end{aligned} \tag{7.33}$$

□

For later purpose we record the variance of the increments, namely,

$$\mathbb{E}\big[(B_{t+h} - B_t)^2\big] = 2k(0) - 2k(h) = |h| - \frac{h^2}{2\pi}. \tag{7.34}$$

Thus, for small time increments we recover the variance of standard Brownian increments. We also record the covariance of two distinct increments, namely, for $h, u \geq 0$ and $t + h \leq s$,

$$\begin{aligned} \mathbb{E}\big[(B_{t+h} - B_t)(B_{s+u} - B_s)\big] &= \frac{1}{4\pi}\Big((s+u-t-h)^2 - (s+u-t)^2 - (s-t-h)^2 + (s-t)^2\Big) \\ &= -\frac{1}{2\pi} hu. \end{aligned} \tag{7.35}$$

Thus, two distinct increments are not independent, however, for $h, u \downarrow 0$ the covariance is negligible compared to the variance of the individual increments (since $hu = o(h) + o(u)$). This will be needed in Lemma 7.5 below.

Exponential moments for the mean-centred Brownian bridge. For $k \in \mathbb{N}$, define the random variables

$$A_k = \frac{k}{\sqrt{\pi}} \int_0^{2\pi} \mathrm{d}t\, B_t \cos(kt), \quad A_k^* = \frac{k}{\sqrt{\pi}} \int_0^{2\pi} \mathrm{d}t\, B_t \sin(kt). \tag{7.36}$$

In view of Lemma 7.3, these random variables are i.i.d. standard normal. They represent the Fourier coefficients of B, i.e.,

$$B_t = \frac{1}{\sqrt{\pi}} \sum_{k\in\mathbb{N}} \Big[\frac{A_k}{k} \cos(kt) + \frac{A_k^*}{k} \sin(kt)\Big], \tag{7.37}$$

where the series converges in $L^2(0, 2\pi)$ $\mathbb{P}$-a.s. The expansion in (7.37) is the Karhunen-Loève expansion of the Gaussian process B (see Alexanderian [1], Deheuvels [10], and references therein).

Lemma 7.4. *For every $-\infty < s < 1$,*

$$\mathbb{E}\left[\exp\left[\tfrac{1}{2}s\left(\int_0^{2\pi} \mathrm{d}t\, B_t^2\right)\right]\right] = \prod_{k\in\mathbb{N}} \Big(1 - \frac{s}{k^2}\Big)^{-1} \quad \mathbb{P}\text{-a.s.} \tag{7.38}$$

For every $-\infty < s < 4$,

$$\mathbb{E}\left[\exp\left[\tfrac{1}{2}s\left(\int_0^{2\pi} \mathrm{d}t\, B_t^2 - A_1^2 - A_1^{*2}\right)\right]\right] = \prod_{k\in\mathbb{N}\setminus\{1\}} \Big(1 - \frac{s}{k^2}\Big)^{-1} \quad \mathbb{P}\text{-a.s.} \tag{7.39}$$

Proof. Note that (7.37) implies

$$\int_0^{2\pi} \mathrm{d}t\, B_t^2 = \sum_{k\in\mathbb{N}} \tfrac{1}{k^2}(A_k^2 + A_k^{*2}). \tag{7.40}$$

Since A_k, A_k^* are i.i.d. standard normal, the claim follows from the identity $\mathbb{E}[\exp(\frac{1}{2}uX^2)] = (1-u)^{-1/2}$ when X is standard normal and $u < 1$. Apply this identity with $u = s/k^2$, $k \in \mathbb{N}$ to get (7.38). The proof of (7.39) is similar. □

The next lemma allows us to subsume the error term $O(\varepsilon)Y_2$ into the error term $O(\varepsilon)N$ appearing in (6.41).

Lemma 7.5. *Let $0 \le t_1 < \cdots < t_n < 2\pi$, $t_{n+1} = t_1 + 2\pi$, and $\theta_i = t_{i+1} - t_i$. For every $s \in (0,1)$,*

$$\mathbb{E}\left[\exp\Big(\tfrac{1}{2}s\sum_{i=1}^{n}\frac{(B_{t_{i+1}} - B_{t_i})^2}{\theta_i}\Big)\right] = \frac{1}{(\sqrt{1-s}\,)^{n-1}}. \tag{7.41}$$

Proof. Define $L_i = (B_{t_{i+1}} - B_{t_i})/\sqrt{\theta_i}$, $1 \le i \le n$. It follows from (7.34) and (7.35) that $(L_i)_{1\le i\le n}$ is a Gaussian vector with covariance matrix $C = (C_{ij})_{1\le i,j\le n}$ given by

$$C_{ij} = \delta_{ij} - \frac{\sqrt{\theta_i\theta_j}}{2\pi}. \tag{7.42}$$

Hence $C = \mathrm{id} - P$, where P is the orthogonal projection onto the linear span of $(\sqrt{\theta_i/2\pi})_{i=1}^n$ in $\mathbb{R}^n$. Thus, C is the orthogonal projection onto the $(n-1)$-dimensional hyperplane defined by $\{\ell = (\ell_i)_{i=1}^n\colon\ \sum_{i=1}^n \ell_i\sqrt{\theta_i} = 0\}$. Using orthonormal coordinates on that hyperplane, we find that

$$\mathbb{E}\left[\exp\Big(\tfrac{1}{2}s\sum_{i=1}^{n}L_i^2\Big)\right] = \frac{1}{\sqrt{2\pi}^{\,n-1}}\int_{\mathbb{R}^{n-1}}\mathrm{d}x\,\exp\Big(\tfrac{1}{2}sx^2 - \tfrac{1}{2}x^2\Big) = \frac{1}{(\sqrt{1-s}\,)^{n-1}}, \tag{7.43}$$

which settles the claim. □

7.3 Discretisation errors

Deterministic discretisation errors. Later we need to bound the error in the approximations

$$\sum_{i=1}^{n}\overline{B}_{t_i}\theta_i \approx \int_0^{2\pi} B_t\,\mathrm{d}t = 0, \qquad \sum_{i=1}^{n}\overline{B}_{t_i}{}^2\theta_i \approx \int_0^{2\pi} B_t^2\,\mathrm{d}t, \tag{7.44}$$

and discretisation errors for Fourier coefficients. Lemmas 7.7 and 7.8 treat the errors as random variables and provide bounds for their exponential moments. The proofs build on bounds for deterministic discretisation errors, which we provide first.

Let $\mathcal{H}$ be the space of absolutely continous functions $f : [0,2\pi] \to \mathbb{R}$ with square-integrable derivative, satisfying $\tau(2\pi) = \tau(0)$ and $\int 0^{2\pi}\tau(s)\mathrm{d}s = 0$. Let $\tau \in \mathcal{H}$. Note that $|\tau(t) - \tau(s)| = |\int_s^t \dot\tau(s')\mathrm{d}s'| \le ||\dot\tau||_2\sqrt{2\pi}$, and hence

$$\begin{aligned}|\tau(t)| &= \left|\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}s\,[\tau(t) - \tau(s)]\right| \le \sqrt{2\pi}\,\|\dot\tau\|_2,\\ \|\tau\|_\infty &\le \sqrt{2\pi}\,\|\dot\tau\|_2.\end{aligned} \tag{7.45}$$

Set

$$\overline{\tau_i} = \tfrac{1}{2}\big[\tau(t_i) + \tau(t_{i+1})\big], \tag{7.46}$$

and consider the sums

$$\begin{aligned}\Lambda_1 &= \sum_{i=1}^{n}\overline{\tau_i}^2\,\theta_i, \quad \Lambda_2 = \sum_{i=1}^{n}\tau(t_i)\,\theta_i,\\ \Lambda_3 &= \sum_{i=1}^{n}\tau(t_i)\,\theta_i\cos t_i, \quad \Lambda_4 = \sum_{i=1}^{n}\tau(t_i)\,\theta_i\sin t_i.\end{aligned} \tag{7.47}$$

Lemma 7.6. *Suppose that $\tau \in \mathcal{H}$ and put $\varepsilon_n = \sum_{i=1}^n \theta_i^3 = y_2(z)$. Then*

$$\begin{aligned}&\big|\Lambda_1 - \|\tau\|_2^2\big| \le \sqrt{\varepsilon_n}\,\|\tau\|_\infty\|\dot\tau\|_2, && \big|\Lambda_2\big| \le \sqrt{\varepsilon_n/3}\,\|\dot\tau\|_2,\\ &\big|\Lambda_3 - \textstyle\int_0^{2\pi}\mathrm{d}t\,\tau(t)\cos t\big| \le 2\sqrt{\varepsilon_n/3}\,\|\dot\tau\|_2, && \big|\Lambda_4 - \textstyle\int_0^{2\pi}\mathrm{d}t\,\tau(t)\sin t\big| \le 2\sqrt{\varepsilon_n/3}\,\|\dot\tau\|_2.\end{aligned} \tag{7.48}$$

Proof. Note that

$$\big|\tau(t)^2-\overline{\tau_i}^2\big| = \big|\big(\tau(t)-\overline{\tau_i}\big)(\tau(t)+\overline{\tau_i})\big| \le 2\|\tau\|_\infty\,|\tau(t)-\overline{\tau_i}| \tag{7.49}$$

and

$$\begin{aligned}&\int_{t_i}^{t_{i+1}}\mathrm{d}t\,|\tau(t)-\overline{\tau_i}| = \int_{t_i}^{t_{i+1}}\mathrm{d}t\left|\int_{t_i}^{t}\mathrm{d}s\,\dot\tau(s)\right| \le \int_{t_i}^{t_{i+1}}\mathrm{d}t\int_{t_i}^{t}\mathrm{d}s\,|\dot\tau(s)| = \int_{t_i}^{t_{i+1}}\mathrm{d}s\,(t_{i+1}-s)\,|\dot\tau(s)|\\ &= \int_{t_i}^{t_{i+1}}\mathrm{d}s\,\tfrac12\big[(t_{i+1}-s)+(s-t_i)\big]\,|\dot\tau(s)| = \tfrac12\theta_i\int_{t_i}^{t_{i+1}}\mathrm{d}s\,|\dot\tau(s)|.\end{aligned} \tag{7.50}$$

It therefore follows that

$$\begin{aligned}&\Big|\Lambda_1-\|\tau\|_2^2\Big| = \left|\sum_{i=1}^n\overline{\tau_i}^2\theta_i-\int_0^{2\pi}\mathrm{d}t\,\tau(t)^2\right| = \left|\sum_{i=1}^n\int_{t_i}^{t_{i+1}}\mathrm{d}t\,\big(\overline{\tau_i}^2-\tau(t)^2\big)\right|\\ &\le 2\|\tau\|_\infty\sum_{i=1}^n\int_{t_i}^{t_{i+1}}\mathrm{d}t\,|\tau(t)-\overline{\tau_i}| \le \|\tau\|_\infty\sum_{i=1}^n\theta_i\int_{t_i}^{t_{i+1}}\mathrm{d}t\,|\dot\tau(t)| = \|\tau\|_\infty\int_0^{2\pi}\mathrm{d}t\,|\dot\tau(t)|\sum_{i=1}^n\theta_i\,\mathbf{1}_{[t_i,t_{i+1})}(t)\\ &\le \|\tau\|_\infty\|\dot\tau\|_2\left(\int_0^{2\pi}\mathrm{d}t\,\left[\sum_{i=1}^n\theta_i\mathbf{1}_{[t_i,t_{i+1})}(t)\right]^2\right)^{1/2} = \|\tau\|_\infty\|\dot\tau\|_2\left(\sum_{i=1}^n\theta_i^3\right)^{1/2},\end{aligned} \tag{7.51}$$

which is the inequality for Λ_1.

Let $f\colon\mathbb{R}\to\mathbb{R}$ be absolutely continuous and 2π-periodic. Suppose for simplicity that $t_1=0$ (otherwise replace integrals over $[0,2\pi]$ by integrals over $[t_1,t_{n+1}]=[t_1,t_1+2\pi]$). By partial integration we have

$$\sum_{i=1}^n\theta_i f(t_i)-\int_0^{2\pi}\mathrm{d}s\,f(s) = \sum_{i=1}^n\int_{t_i}^{t_{i+1}}\mathrm{d}s\,[f(t_i)-f(s)] = -\sum_{i=1}^n\int_{t_i}^{t_{i+1}}\mathrm{d}u\,(t_{i+1}-u)\dot f(u), \tag{7.52}$$

from which we get, by Cauchy-Schwarz,

$$\left|\sum_{i=1}^n\theta_i f(t_i)-\int_0^{2\pi}\mathrm{d}s\,f(s)\right| \le \|\dot f\|_2\left(\sum_{i=1}^n\tfrac13\theta_i^3\right)^{1/2} = \|\dot f\|_2\sqrt{\varepsilon_n/3}. \tag{7.53}$$

The bound on Λ_2 is obtained from (7.53) by picking $f(s)=\tau(s)$ and using that $\int_0^{2\pi}\mathrm{d}s\,\tau(s)=0$. The bounds on Λ_3 and Λ_4 are obtained from (7.53) by picking $f(s)=\cos s$ and $f(s)=\sin s$, respectively, and using that $\int_0^{2\pi}\mathrm{d}s\,\cos s=\int_0^{2\pi}\mathrm{d}s\,\sin s=0$. □

Random discretisation errors. The estimates of *deterministic* discretisation errors in Lemma 7.6 can be used to derive estimates of exponential moments of *random* discretisation errors which is needed later.

Lemma 7.7 (Discretised mean)**.** *Put $\varepsilon_n=\sum_{i=1}^n\theta_i^3$. Then, for every $s\in\mathbb{R}$,*

$$\mathbb{E}\left[\exp\left(s\beta^{1/2}\sum_{i=1}^n\overline{B_{t_i}}\theta_i\right)\right] \le \exp\Big(\tfrac{1}{32\pi}s^2\beta\varepsilon_n\Big). \tag{7.54}$$

Proof. Write

$$\mathbb{E}\left[\exp\left(s\beta^{1/2}\sum_{i=1}^n\overline{B_{t_i}}\,\theta_i\right)\right] = \mathbb{E}\left[\exp\left(2\pi s\beta^{1/2}\int_0^{2\pi}\Gamma(\mathrm{d}t)B_t\right)\right], \tag{7.55}$$

where

$$\Gamma = \sum_{i=1}^n\tfrac12\big(\delta_{t_i}+\delta_{t_{i+1}}\big)\frac{\theta_i}{2\pi} = \sum_{i=1}^n\delta_{t_i}\,\frac{t_{i+1}-t_{i-1}}{4\pi} \tag{7.56}$$

is a probability measure on $[0,2\pi]$. Apply (7.29) to get

$$\mathbb{E}\left[\exp\left(2\pi s\beta^{1/2}\int_0^{2\pi}\Gamma(\mathrm{d}t)B_t\right)\right]=\exp\left(\tfrac12(2\pi s)^2\beta G\right) \tag{7.57}$$

with

$$G=\int_0^{2\pi}\Gamma(\mathrm{d}s)\int_0^{2\pi}\Gamma(\mathrm{d}t)\,k(s-t)=\frac{1}{(4\pi)^2}\sum_{i=1}^n(t_{i+1}-t_{i-1})\sum_{j=1}^n(t_{j+1}-t_{j-1})\,k(t_i-t_j). \tag{7.58}$$

The proof proceeds in two approximation steps. First, use (7.32) to write

$$\begin{aligned}\sum_{i=1}^n(t_{i+1}-t_{i-1})\,k(t_i-t_j)&=\sum_{i=1}^n\int_{t_i}^{t_{i+1}}\mathrm{d}s\,\big[k(t_i-t_j)+k(t_{i+1}-t_j)-2k(s-t_j)\big]\\&=-\sum_{i=1}^n\int_{t_i}^{t_{i+1}}\mathrm{d}s\int_{t_i-t_j}^{s-t_j}\mathrm{d}u\,\dot k(u)+\sum_{i=1}^n\int_{t_i}^{t_{i+1}}\mathrm{d}s\int_{s-t_j}^{t_{i+1}-t_j}\mathrm{d}u\,\dot k(u)\\&=-\sum_{i=1}^n\int_{t_i-t_j}^{t_{i+1}-t_j}\mathrm{d}u\,\dot k(u)\int_{u+t_j}^{t_{i+1}}\mathrm{d}s+\sum_{i=1}^n\int_{t_i-t_j}^{t_{i+1}-t_j}\mathrm{d}u\,\dot k(u)\int_{t_i}^{t_j+u}\mathrm{d}s\\&=\sum_{i=1}^n\int_{t_i}^{t_{i+1}}\mathrm{d}u'\,\dot k(u'-t_j)\,(2u'-t_i-t_{i+1}).\end{aligned} \tag{7.59}$$

Substitute this into (7.58) to get

$$G=\frac{1}{(4\pi)^2}\sum_{i=1}\int_{t_i}^{t_{i+1}}\mathrm{d}u'\,(2u'-t_i-t_{i+1})\sum_{j=1}^n(t_{j+1}-t_{j-1})\,\dot k(u'-t_j). \tag{7.60}$$

Next, use (7.32) to write

$$\begin{aligned}\sum_{j=1}^n(t_{j+1}-t_{j-1})\,\dot k(u'-t_j)&=\sum_{j=1}^n\int_{t_j}^{t_{j+1}}\mathrm{d}s\,\big[\dot k(u'-t_j)+\dot k(u'-t_{j+1})-2\dot k(u'-s)\big]\\&=\sum_{j=1}^n\int_{t_j}^{t_{j+1}}\mathrm{d}s\int_{u'-s}^{u'-t_j}\mathrm{d}u\,\ddot k(u)-\sum_{j=1}^n\int_{t_j}^{t_{j+1}}\mathrm{d}s\int_{u'-t_{j+1}}^{u'-s}\mathrm{d}u\,\ddot k(u)\\&=\sum_{j=1}^n\int_{u'-t_{j+1}}^{u'-t_j}\mathrm{d}u\,\ddot k(u)\int_{u'-u}^{t_{j+1}}\mathrm{d}s-\sum_{j=1}^n\int_{u'-t_{j+1}}^{u'-t_j}\mathrm{d}u\,\ddot k(u)\int_{t_j}^{u'-u}\mathrm{d}s\\&=\sum_{j=1}^n\int_{u'-t_{j+1}}^{u'-t_j}\mathrm{d}u\,\ddot k(u)\,(2u-2u'-t_j+t_{j+1}).\end{aligned} \tag{7.61}$$

Substitute this into (7.60) to get

$$G=\frac{1}{(4\pi)^2}\sum_{i=1}^n\int_{t_i}^{t_{i+1}}\mathrm{d}u'\,(2u'-t_i-t_{i+1})\sum_{j=1}^n\int_{t_j}^{t_{j+1}}\mathrm{d}u''\,(t_{j+1}-2u''-t_j)\;\ddot k(u'-u''). \tag{7.62}$$

Finally, note that $|2u'-t_i-t_{i+1}|\le\theta_i$, $|t_{j+1}-2u''-t_j|\le\theta_j$ and $\|\ddot k\|_\infty=\frac12$, to estimate

$$\begin{aligned}|G|&\le\frac18\left[\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}t\left(\sum_{i=1}^n\theta_i1_{[t_i,t_{i+1})}(t)\right)\right]^2\le\frac18\,\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}t\,\left[\sum_{i=1}^n\theta_i1_{[t_i,t_{i+1})}(t)\right]^2\\&=\frac18\,\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}t\,\sum_{i=1}^n\theta_i^21_{[t_i,t_{i+1})}(t)=\frac18\,\frac{1}{2\pi}\sum_{i=1}^n\theta_i^3.\end{aligned} \tag{7.63}$$

Combine this with (7.55) and (7.57), and note that $G\ge0$, to get the claim in (7.54). $\square$

Lemma 7.8. *Put $\varepsilon_n = \sum_{i=1}^n \theta_i^3$. Then, for every $s \in \mathbb{R}$ with $|s| < 1/\sqrt{2\pi\varepsilon_n}$,*

$$\mathbb{E}\left[\exp\left[\tfrac{1}{2}s\left(\sum_{i=1}^n \overline{B_{t_i}}^2\theta_i - \int_0^{2\pi} \mathrm{d}t\, B_t^2\right)\right]\right] \leq \exp\Big(|s|\, O\big(\sqrt{\varepsilon_n}\log(1/\varepsilon_n)\big)\Big). \tag{7.64}$$

Proof. The proof uses a determinant formula for exponent moments of quadratic functionals in abstract Wiener spaces (see (7.73) below).

We view B as a random variable taking values in the Banach space E of mean-zero, 2π-periodic, continuous functions $f\colon \mathbb{R} \to \mathbb{R}$, equipped with the supremum norm $\|\cdot\|_\infty$. The scalar product $\langle f, g\rangle_{\mathcal{H}} = \int_0^{2\pi} \mathrm{d}t\, f'(t)g'(t)$ turns $\mathcal{H} \subset E$ into a real Hilbert space. For μ a finite signed measure on $[0, 2\pi]$ with total mass zero, define $(-\Delta)^{-1}\mu \in \mathcal{H}$ by

$$(-\Delta)^{-1}\mu(t) = \int_0^{2\pi} \mathrm{d}\mu(t')\, k(t-t'), \qquad t \in \mathbb{R}, \tag{7.65}$$

with $k(t-t')$ the Green function from Lemma 7.3. Following the proof of Lemma 7.3(b), we can check that $g = (-\Delta)^{-1}\mu$ is in $\mathcal{H}$ and satisfies $-g'' = \mu$ in a distributive sense:

$$\mu(f) = \int_0^{2\pi} \mathrm{d}\mu\, f = \int_0^{2\pi} \mathrm{d}t\, f(t)(-g''(t)) = \langle g, f\rangle_{\mathcal{H}} = \langle(-\Delta)^{-1}\mu, f\rangle_{\mathcal{H}}, \qquad f \in \mathcal{H}. \tag{7.66}$$

When $f \in E \setminus \mathcal{H}$, we define $\langle g, f\rangle_{\mathcal{H}}$ as $\mu(f)$, so that (7.66) remains true. A slight adaptation of the proof of Lemma 7.3(b) then shows that

$$\mathbb{E}\big[\mathrm{e}^{\mathrm{i}\langle g, B\rangle_{\mathcal{H}}}\big] = \mathbb{E}\big[\mathrm{e}^{\mathrm{i}\mu(B)}\big] = \mathrm{e}^{-\frac{1}{2}\int_0^{2\pi}\mathrm{d}\mu(t)\, g(t)} = \mathrm{e}^{-\frac{1}{2}\|g\|_{\mathcal{H}}^2}. \tag{7.67}$$

Now consider the continuous quadratic form

$$Q(f,f) = \sum_{i=1}^n \overline{f(t_i)}^2\theta_i - \int_0^{2\pi} \mathrm{d}t\, f(t)^2, \qquad f \in E. \tag{7.68}$$

The restriction of Q to $\mathcal{H}$ is represented by a bounded symmetric operator $\tilde{Q}\colon \mathcal{H} \to \mathcal{H}$ that is uniquely defined by the requirement that

$$Q(h,h) = \langle h, \tilde{Q}h\rangle_{\mathcal{H}}, \qquad h \in \mathcal{H}. \tag{7.69}$$

From (7.45) and the first line of (7.48) we know that

$$\big|Q(h,h)\big| \leq \sqrt{\varepsilon_n}\,\|h\|_\infty\|\dot{h}\|_2 \leq \sqrt{2\pi\varepsilon_n}\,\|\dot{h}\|_2^2 = \sqrt{2\pi\varepsilon_n}\,\|h\|_{\mathcal{H}}^2, \qquad h \in \mathcal{H}. \tag{7.70}$$

Hence $\|\tilde{Q}\| \leq \sqrt{2\pi\varepsilon_n}$ and so, for all $s \in \mathbb{R}$ with $|s| \leq 1/\sqrt{2\pi\varepsilon_n}$, the operator $\mathrm{id} - s\tilde{Q}$ is positive definite and invertible in $\mathcal{H}$, with bounded inverse.

Next, we check that $\tilde{Q}$ is a trace-class operator (Simon [37]). From(7.68) we see that $\tilde{Q}$ is a difference of two terms. The first term corresponds to the finite sum in (7.68), has finite rank, and is therefore trivially trace class. The second term is $(-\Delta)^{-1}$. Indeed, for $f \in \mathcal{H}$, setting $F = (-\Delta)^{-1}f \in \mathcal{H}$ and integrating by part, we have

$$\int_0^{2\pi} \mathrm{d}t\, f(t)^2 = \int_0^{2\pi} \mathrm{d}t\, f(t)(-F''(t)) = \int_0^{2\pi} \mathrm{d}t\, f'(t)F'(t) = \langle f, (-\Delta)^{-1}f\rangle_{\mathcal{H}}. \tag{7.71}$$

Consider the orthonormal basis of $\mathcal{H}$ consisting of the vectors

$$e_k(t) = \frac{\cos(kt)}{k\sqrt{\pi}}, \quad e_k^*(t) = \frac{\sin(kt)}{k\sqrt{\pi}}, \qquad k \in \mathbb{N}. \tag{7.72}$$

Then $(-\Delta)^{-1}e_k = \frac{1}{k^2}e_k$ and $(-\Delta)^{-1}e_k^* = \frac{1}{k^2}e_k^*$. Thus, $(-\Delta)^{-1}$ is trace class with trace $2\sum_{k=1}^\infty 1/k^2 < \infty$, and $\tilde{Q}$, as a difference of two trace class operators, is itself trace class. It follows from general results on abstract Wiener spaces (Chiang, Chow and Lee [5]) that, for every $s \in \mathbb{R}$ with $|s| < (2\pi\varepsilon_n)^{-1}$, the exponential moments of $Q(B,B)$ are given by

$$\mathbb{E}\Big[\exp\big[\tfrac{1}{2}sQ(B,B)\big]\Big] = \det(\mathrm{id} - s\tilde{Q})^{-1/2}, \tag{7.73}$$

where the determinant is a Fredholm determinant. Indeed, if $\lambda_1 \geq \lambda_2 \geq \cdots$ are the eigenvalues of $\tilde{Q}$, then $\max_{j\in\mathbb{N}} |\lambda_j| = \|\tilde{Q}\| = O(\sqrt{\varepsilon_n})$, and the Fredholm determinant is given by

$$\det(\mathrm{id} - s\tilde{Q}) = \prod_{j\in\mathbb{N}} (1 - |s|\lambda_j) = \exp\left[|s|\, O\Big(\sum_{j\in\mathbb{N}} \lambda_j\Big)\right] = \exp\Big(O(|s|\,\mathrm{Tr}\,\tilde{Q})\Big). \tag{7.74}$$

Thus, it remains to estimate the trace of $\tilde{Q}$. To that aim we apply the first line of (7.48) for $\tau(t) = e_k(t)$. Since $\|e_k\|_2 = 1/k$ and $\|\dot{e}_k\|_\infty = 1/\sqrt{\pi}$, this gives

$$\big|\langle e_k, \tilde{Q}e_k\rangle_{\mathcal{H}}\big| = \big|Q(e_k,e_k)\big| \leq \sqrt{\varepsilon_n}\,\|e_k\|_\infty \|\dot{e}_k\|_2 = \frac{\sqrt{\varepsilon_n/\pi}}{k}. \tag{7.75}$$

On the other hand, clearly $Q(e_k,e_k) \leq 4\pi\|e_k\|_\infty^2 = 4/k^2$. Similar bounds hold for $Q(e_k^*, e_k^*)$. Therefore

$$\big|\mathrm{Tr}\,\tilde{Q}\big| \leq 2\sum_{k\in\mathbb{N}} \min\big((\sqrt{\varepsilon_n/\pi})/k, 4/k^2\big) = O\big(\sqrt{\varepsilon_n}\log(1/\varepsilon_n)\big). \tag{7.76}$$

The proof is concluded with (7.73) and (7.74). □

Fourier coefficients. Define the discretized Fourier coefficients

$$D_1 = \frac{1}{\sqrt{\pi}}\sum_{i=1}^n \theta_i\,\overline{B_{t_i}\cos t_i}, \qquad D_1^* = \frac{1}{\sqrt{\pi}}\sum_{i=1}^n \theta_i\,\overline{B_{t_i}\sin t_i}. \tag{7.77}$$

By a slight abuse of notation we shall use the same letter D_1, D_1^* for the random variable obtained by substituting $t_i \to T_i$, $\theta_i \to \Theta_i$.

Lemma 7.9. *Put $\varepsilon_n = \sum_{i=1}^n \theta_i^3$. Then, for every $|s| < 1/8\sqrt{2\pi\varepsilon_n/3}$,*

$$\mathbb{E}\Big[\exp\Big(\tfrac{1}{2}s[A_1^2 - D_1^2]\Big)\Big] \leq \exp\Big(|s|\,O\big(\sqrt{\varepsilon_n}\log(1/\varepsilon_n)\big)\Big). \tag{7.78}$$

A similar bound holds for $A_1^{*2} - D_1^{*2}$.

Proof. The proof is similar to that of Lemma 7.8. Let

$$Q(f,f) = \Big(\frac{1}{\sqrt{\pi}}\sum_{i=1}^n \theta_i\,\overline{f(t_i)\cos t_i}\Big)^2 - \Big(\frac{1}{\sqrt{\pi}}\int_0^{2\pi} \mathrm{d}t\, f(t)\,\cos t\Big)^2 \tag{7.79}$$

and let $\tilde{Q}\colon \mathcal{H} \to \mathcal{H}$ be the associated symmetric operator. Then $\tilde{Q}$ has finite rank and is therefore trace-class. Using the identity $|x^2 - y^2| = |x+y|\,|x-y|$, $x, y \in \mathbb{R}$, in combination with (7.45) and the second line of (7.48), we see that for all $h \in \mathcal{H}$,

$$|Q(h,h)| \leq 4\|h\|_\infty \left|\sum_{i=1}^n \theta_i\,\overline{h(t_i)\cos t_i} - \int_0^{2\pi} \mathrm{d}t\, h(t)\,\cos t\right| \leq 8\sqrt{\varepsilon_n/3}\,\|h\|_\infty\,\|\dot{h}\|_2 \leq 8\sqrt{2\pi\varepsilon_n/3}\,\|h\|_{\mathcal{H}}^2. \tag{7.80}$$

Consequently, $\|\tilde{Q}\| \leq 8\sqrt{2\pi\varepsilon_n/3}$, $\mathrm{id} - s\tilde{Q}$ is positive definite for all $|s| < 1/8\sqrt{2\pi\varepsilon_n/3}$, and

$$|Q(e_k,e_k)| \leq 8\sqrt{2\pi\varepsilon_n/3}\,\|e_k\|_\infty\,\|\dot{e}_k\|_\infty = 8\sqrt{2\varepsilon_n/3}/k = O(\sqrt{\varepsilon_n}/k), \tag{7.81}$$

and similarly for e_k^*. The proof is concluded in the same way as that of Lemma 7.8. □

8 Asymptotics of surface integrals: proof of moderate deviations

In this section we collect the preparations in Sections 5–7 and prove Theorems 2.5–2.6. Section 8.1 proves an upper bound for the key surface integral $I^{\mathrm{UB}}(\kappa,\beta;C,\varepsilon)$ (Proposition 8.1), which together with conditions (C1) and (C3) in Section 2.2 proves the desired upper bound in Theorem 2.5. The proof consists of a sequence of steps involving decomposition, elimination and separation of terms. A particularly delicate point is how to control the integral over m in (6.41): we show that only small values of m contribute, namely, $|m| = O(\beta^{1/6})$. Section 8.2 proves a lower bound for the key surface integral $I^{\mathrm{LB}}(\kappa,\beta;C,\varepsilon)$ (Proposition 8.2), which together with conditions (C1) and (C2) in Section 2.2 proves the desired lower bound in Theorem 2.5. In Sections 8.1–2.2 we also prove Theorem 2.6. The proof does not need conditions (C1)–(C3), but has non-matching upper and lower bound. The proof of the lower bound relies on a rough version of (C1) that can be proved easily.

8.1 Upper bound

We prove the following:

Proposition 8.1 (Upper bound key integral)**.** *For every $C \in (0,\infty)$ and all $\varepsilon > 0$ sufficiently small, there exists a $c = c(C,\varepsilon) > 0$ such that*

$$\begin{aligned}\limsup_{\beta\to\infty} & \frac{1}{\beta^{1/3}} \log \mathcal{I}^{\mathrm{UB}}(\kappa,\beta;C,\varepsilon) \\ & \leq 2\pi G_\kappa \tau_* + \limsup_{\beta\to\infty} \sup_{|m|\leq c\beta^{1/6}} \frac{1}{\beta^{1/3}} \log \widehat{\mathbb{E}}\Big[\exp\big(\tfrac{1}{2}[1+O(\varepsilon)]\chi\big)\, \mathbf{1}_{\{Z^{(m)}\in\mathcal{O}\}}\Big] + O(\varepsilon),\end{aligned} \tag{8.1}$$

where $\chi = \sum_{i=1}^N \Theta_i \overline{B_{T_i}}^2 - D_1^2 - D_1^{*2}$ and D_1, D_1^* are defined in (7.77).

Before we embark on the proof of the proposition, we use it to complete the proofs of the upper bounds in Theorems 2.5 and 2.6.

Proof of the upper bound in Theorem 2.5. Proposition 8.1, when combined with conditions (C1) and (C3) in Theorem 2.5, yields

$$\limsup_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log \mathcal{I}^{\mathrm{UB}}(\kappa,\beta;C,\varepsilon) \leq 2\pi G_\kappa(\tau_* - \tau_{**}) + O(\varepsilon). \tag{8.2}$$

with τ_{**} given in (2.29). Combining with Corollary 6.4, we get

$$\limsup_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log\Big(\mathrm{e}^{\beta\, I^*(\pi R_c^2)} \mu_\beta\Big(\mathcal{V}_{C\beta^{-2/3}} \cap \mathcal{D}_\varepsilon(0) \cap \mathcal{C}_{C\beta^{-2/3}}(0)\Big)\Big) \leq 2\pi G_\kappa(\tau_* - \tau_{**}) + O(\varepsilon). \tag{8.3}$$

Combining with Lemma 6.2, we find

$$\limsup_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log\Big(\mathrm{e}^{\beta\, I^*(\pi R_c^2)} \mu_\beta\Big(\mathcal{V}_{C\beta^{-2/3}} \cap \mathcal{D}_\varepsilon\Big)\Big) \leq 2\pi G_\kappa(\tau_* - \tau_{**}) + O(\varepsilon). \tag{8.4}$$

Contributions from $V_{C\beta^{-2/3}} \cap \mathcal{D}_\varepsilon$ are bounded by Lemma 6.1, and are negligible. The upper bound in Theorem 2.5 follows after letting $\varepsilon \downarrow 0$ with C fixed. $\square$

Proof of the upper bound in Theorem 2.6. Without the conditions (C1)–(C3) from Theorem 2.5, we estimate the right-hand side of (8.1) by dropping the indicator. We decompose χ into three parts

$$\chi = E_1 + E_2 + E_3, \tag{8.5}$$

with

$$\begin{aligned} E_1 &= \sum_{i=1}^{N} \overline{B_{T_i}}^2 \Theta_i - \int_0^{2\pi} \mathrm{d}t\, B_t^2, \\ E_2 &= \int_0^{2\pi} \mathrm{d}t\, B_t^2 - A_1^2 - A_1^{*2}, \\ E_3 &= A_1^2 + A_1^{*2} - D_1^2 - D_1^{*2}. \end{aligned} \tag{8.6}$$

We first look at conditional expectations. Note that Y_1 depends on N and Θ_i, $1 \le i \le n$, alone. A repeated application of the Cauchy-Schwarz inequality yields

$$\begin{aligned} &\widehat{\mathbb{E}}\Big[\mathrm{e}^{\frac{1}{2}[1+O(\varepsilon)]\chi} \mid N, (\Theta_i)_{i=1}^N\Big] \\ &\le \widehat{\mathbb{E}}\Big[\mathrm{e}^{\frac{3}{2}[1+O(\varepsilon)]E_1} \mid N, (\Theta_i)_{i=1}^N\Big]^{1/3} \widehat{\mathbb{E}}\Big[\mathrm{e}^{\frac{3}{2}[1+O(\varepsilon)]E_2} \mid N, (\Theta_i)_{i=1}^N\Big]^{1/3} \widehat{\mathbb{E}}\Big[\mathrm{e}^{\frac{3}{2}[1+O(\varepsilon)]E_3} \mid N, (\Theta_i)_{i=1}^N\Big]^{1/3}. \end{aligned} \tag{8.7}$$

By Lemma 7.8, for every $s \in (-1,1)$ with $|s| < 1/\sqrt{2\pi Y_1}$,

$$\log \widehat{\mathbb{E}}\Big[\mathrm{e}^{\frac{1}{2}s\, E_1} \,\Big|\, N, (\Theta_i)_{i=1}^N\Big] \le |s|\sqrt{Y_1}\log(1/Y_1) \quad \widehat{\mathbb{P}}\text{-a.s.} \tag{8.8}$$

By Lemma 7.9, for every $|s| \le 1/\sqrt{2\pi Y_1}$,

$$\log \widehat{\mathbb{E}}\Big[\mathrm{e}^{\frac{1}{2}s(A_1^2 - D_1^2)} \,\Big|\, N, (\Theta_i)_{i=1}^N\Big] \le |s|\sqrt{Y_1}\log(1/Y_1) \quad \widehat{\mathbb{P}}\text{-a.s.} \tag{8.9}$$

A similar estimate holds for $A_1^{*2} - D_1^{*2}$. Via Cauchy-Schwarz, it follows that

$$\log \widehat{\mathbb{E}}\Big[\mathrm{e}^{\frac{1}{2}sE_3} \,\Big|\, N, (\Theta_i)_{i=1}^N\Big] \le 2|s|\sqrt{Y_1}\log(1/Y_1) \quad \widehat{\mathbb{P}}\text{-a.s.} \tag{8.10}$$

as long as $2|s| \le 1/\sqrt{2\pi Y_1}$. We would like to apply the estimates in (8.8) and (8.10) with $s = 3[1+O(\varepsilon)]$. From the a priori estimates in Corollary 5.4 we know that $Y_1 = O(\varepsilon)$. Hence $1/\sqrt{2\pi Y_1} \ge c'/\sqrt{\varepsilon}$ for some $c' > 0$. Thus, $|s| \le c'/\sqrt{4\varepsilon}$ is sufficient to ensure the condition $2|s| < 1/\sqrt{2\pi Y_1}$. Therefore we see that $s = 3[1+O(\varepsilon)]$ satisfies the bound $2|s| \le 1/\sqrt{2\pi Y_1}$, so that (8.8) and (8.10) are valid.

In order to get rid of Y_1 in the right-hand side of (8.10), we use two estimates: the a priori estimate $Y_1 = O(\varepsilon)$ and the bound $Y_1 \ge N(2\pi/N)^3$, which gives $\log(1/Y_1) = O(\log N) = O(N)$. Therefore

$$\log \widehat{\mathbb{E}}\Big[\mathrm{e}^{\frac{1}{2}sE_3} \,\Big|\, N, (\Theta_i)_{i=1}^N\Big] \le O(\varepsilon \log N) = O(\varepsilon N) \quad \widehat{\mathbb{P}}\text{-a.s.} \tag{8.11}$$

A similar bound holds for E_1. We now estimate the term with E_2. The tilt by $\mathrm{e}^{Y_0 - \beta C_1 Y_1}$ affects the angular point process only, so it is still true under $\widehat{\mathbb{P}}$ that the distribution of $(B_t)_{t\in[0,2\pi]}$ is a mean-centred Brownian bridge independent from N and the Θ_i, $1 \le i \le N$. Since $s = 3[1+O(\varepsilon)] < 4$, by picking ε small enough, we get from Lemma 7.4 that

$$\log \widehat{\mathbb{E}}\Big[\mathrm{e}^{\frac{3}{2}[1+O(\varepsilon)]E_2}\Big] = -\sum_{k=2}^{\infty} \log\Big(1 - \frac{3}{k^2}[1+O(\varepsilon)]\Big) = O(1). \tag{8.12}$$

Note that this upper bound does not grow with β. Hence, combining (8.8), (8.12) and (8.11), inserting into (8.7) and taking expectations, we find

$$\log \widehat{\mathbb{E}}\Big[\mathrm{e}^{\frac{1}{2}[1+O(\varepsilon)]\chi}\Big] \le \log \widehat{\mathbb{E}}\Big[\mathrm{e}^{O(\varepsilon)N}\Big] \quad \widehat{\mathbb{P}}\text{-a.s.} \tag{8.13}$$

Dividing by $\beta^{1/3}$ and taking the limit $\beta \to \infty$, we obtain via Lemma 7.2 that

$$\limsup_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log \mathcal{I}^{\mathrm{UB}}(\kappa, \beta; C, \varepsilon) \le 2\pi G_\kappa \tau_* + O(\varepsilon). \tag{8.14}$$

From here on the upper bound is proven in the same way as in Theorem 2.5. $\square$

Next we turn to the proof of Proposition 8.1. The representation (6.41) for the key integral and the asymptotics from Proposition 7.1 leave us with the task of bounding

$$\limsup_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log\left(\int_{\mathbb{R}} \mathrm{d}m\, \widehat{\mathbb{E}}\Big[\mathrm{e}^{O(\varepsilon)(\beta Y_1 + Y_2 + N) + \frac12 Y_3^{(m)}} \mathbf{1}_{\{Z^{(m)} \in \mathcal{V}'_{C\beta^{-2/3}} \cap \mathcal{C}_{C\beta^{-2/3}} \cap \mathcal{D}'_\varepsilon(0)\}} \Big] \right). \tag{8.15}$$

The main idea is the following. The error terms $O(\varepsilon)(Y_1 + Y_3 + N)$ should be negligible, and therefore our primary concern is $\frac12 Y_3^{(m)}$ in the exponential. In order to deal with this term, we approximate

$$Y_3^{(m)} = \sum_{i=1}^N (m + \overline{B}_{T_i})^2 \Theta_i \approx \int_0^{2\pi} (m + B_t)^2 \mathrm{d}t = 2\pi m^2 + \int_0^{2\pi} B_t^2 \mathrm{d}t \tag{8.16}$$

(recall that $\int_0^{2\pi} B_t \mathrm{d}t = 0$). The resulting expression is problematic because

$$\int_{\mathbb{R}} \mathrm{d}m\, \mathrm{e}^{\pi m^2} = \infty, \quad \mathbb{E}\Big[\mathrm{e}^{\frac12 \int_0^{2\pi} B_t^2 \mathrm{d}t} \Big] = \infty. \tag{8.17}$$

To cure the divergence, we use the geometric constraints. Roughly speaking, we show that the volume constraint imposes that the only relevant contributions are from $|m| = O(\beta^{1/6})$. In addition, we show that the centring constraint imposes that the Fourier coefficients D_1 and D_1^* are negligible, so that we may replace $\sum_{i=1}^N \overline{B_{T_i}}^2 \Theta_i$ by $\chi = \sum_{i=1}^N \overline{B_{T_i}}^2 \Theta_i - D_1^2 - D_1^{*2}$. This helps because of (2.33) in condition (C3).

The proof comes in 8 steps.

1. Decomposition of $Y_3^{(m)}$. As a preliminary step we decompose $Y_3^{(m)}$ as

$$Y_3^{(m)} = \mathcal{E}_1^{(m)} + \mathcal{E}_2 - \mathcal{E}_3 + \chi \tag{8.18}$$

with (recall that $\overline{B_{T_i}} = \frac12 (B_{T_i} + B_{T_{i+1}})$ and χ is defined in (2.32))

$$\mathcal{E}_1^{(m)} = \frac{1}{2\pi} \left(\sum_{i=1}^N (m + \overline{B_{T_i}}) \Theta_i \right)^2, \quad \mathcal{E}_2 = D_1^2 + D_1^{*2}, \quad \mathcal{E}_3 = \frac{1}{2\pi} \left(\sum_{i=1}^N \overline{B_{T_i}} \Theta_i \right)^2. \tag{8.19}$$

To check (8.18), we apply the variance formula

$$\sum_{i=1}^n \left(x_i - \sum_{j=1}^n x_j p_j \right)^2 p_i = \sum_{i=1}^n x_i^2 p_i - \left(\sum_{j=1}^n x_j p_j \right)^2 \tag{8.20}$$

with $n = N$, $p_i = \frac{\Theta_i}{2\pi}$ and $x_i = m + \overline{B_{T_i}}$, respectively, $x_i = \overline{B_{T_i}}$, to get

$$\begin{aligned} Y_3^{(m)} = \sum_{i=1}^N (m + \overline{B_{T_i}})^2 \Theta_i &= \sum_{i=1}^N \left(\overline{B_{T_i}} - \frac{1}{2\pi} \sum_{j=1}^N \overline{B_{T_j}} \Theta_j \right)^2 \Theta_i + \mathcal{E}_1^{(m)} \\ &= \sum_{i=1}^N \overline{B_{T_i}}^2 \Theta_i - \mathcal{E}_3 + \mathcal{E}_1^{(m)}. \end{aligned} \tag{8.21}$$

The claim in (8.18) now follows from the relation

$$\sum_{i=1}^N \overline{B_{T_i}}^2 \Theta_i = \mathcal{E}_2 + \chi. \tag{8.22}$$

Note that $\mathcal{E}_1^{(m)}$, $\mathcal{E}_2$, $\mathcal{E}_3$ are non-negative, while χ is not necessarily so. The terms $\mathcal{E}_1^{(m)}$ and $\mathcal{E}_2$ will be taken care of in Steps 2 and 4 via the volume and centreing constraints. The term χ will be taken care of via (2.32) in condition (C3). The non-negative term $\mathcal{E}_3$ can simply be dropped for the upper bound.

2. Elimination of $\mathcal{E}_1^{(m)}$ with the help of the volume constraint. Next we exploit the volume constraint and the a priori estimates to get rid of $\mathcal{E}_1^{(m)}$. By Proposition 5.8, recalling the definitions in Proposition 6.6, we have

$$\begin{aligned} |S(Z^{(m)})| - \pi R_c^2 &= -C_1^\varepsilon Y_1 + \frac{1+O(\varepsilon)}{2\beta} Y_2 + \frac{1}{2}\frac{1}{(\kappa-1)\beta} Y_3^{(m)} + R_c \sum_{i=1}^N \chi_i^{(m)} \Theta_i \\ &= -C_1^\varepsilon Y_1 + \frac{1+O(\varepsilon)}{2\beta} Y_2 + \frac{1}{2}\sum_{i=1}^N \big(\chi_i^{(m)} + R_c\big)^2 \Theta_i - \pi R_c^2, \end{aligned} \tag{8.23}$$

where we abbreviate

$$\chi_i^{(m)} = \frac{m + \overline{B_{T_i}}}{\sqrt{(\kappa-1)\beta}}. \tag{8.24}$$

By the triangle inequality, on the event that $||S(Z^{(m)})| - \pi R_c^2| \le C\beta^{-2/3}$ we have

$$\left| \frac{1}{2}\sum_{i=1}^N \big(\chi_i^{(m)} + R_c\big)^2 \Theta_i - \pi R_c^2 \right| \le C\beta^{-2/3} + C_1^\varepsilon Y_1 + \frac{1+O(\varepsilon)}{2\beta} Y_2. \tag{8.25}$$

An elementary computation based again on the variance formula in (8.20) gives

$$\begin{aligned} \sum_{i=1}^N \big(\chi_i^{(m)} + R_c\big)^2 \Theta_i &= 2\pi \left(R_c + \sum_{i=1}^N \chi_i^{(m)} \frac{\Theta_i}{2\pi} \right)^2 + \frac{1}{(\kappa-1)\beta} \sum_{i=1}^N \left(\overline{B_{T_i}} - \frac{1}{2\pi}\sum_{j=1}^N \overline{B_{T_j}} \Theta_j \right)^2 \Theta_i \\ &= 2\pi \left(R_c + \sum_{i=1}^N \chi_i^{(m)} \frac{\Theta_i}{2\pi} \right)^2 + \frac{1}{(\kappa-1)\beta} \big(\mathcal{E}_2 + \chi - \mathcal{E}_3\big). \end{aligned} \tag{8.26}$$

Consequently, (8.25) becomes

$$\left| \pi \left(R_c + \sum_{i=1}^N \chi_i^{(m)} \frac{\Theta_i}{2\pi} \right)^2 - \pi R_c^2 \right| \le C\beta^{-2/3} + C_1^\varepsilon Y_1 + \frac{1+O(\varepsilon)}{2\beta} Y_2 + \frac{1}{2(\kappa-1)\beta}\big(\mathcal{E}_2 + \chi - \mathcal{E}_3\big). \tag{8.27}$$

From the a priori estimate in Proposition 5.2 we have $\rho_i = O(\varepsilon)$ and

$$\left| \sum_{i=1}^N \chi_i^{(m)} \frac{\Theta_i}{2\pi} \right| = O(\varepsilon). \tag{8.28}$$

Therefore

$$\pi \left(R_c + \sum_{i=1}^N \chi_i^{(m)} \frac{\Theta_i}{2\pi} \right)^2 - \pi R_c^2 = [2\pi R_c + O(\varepsilon)] \sum_{i=1}^N \chi_i^{(m)} \frac{\Theta_i}{2\pi}, \tag{8.29}$$

which together with (8.27) gives

$$\beta^{-1/2} \sum_{i=1}^N (m + \overline{B_{T_i}})\, \Theta_i \le O(1)\Big(\beta^{-2/3} + Y_1 + \beta^{-1} Y_2 + \beta^{-1}\big(\mathcal{E}_2 + \chi - \mathcal{E}_3\big)\Big). \tag{8.30}$$

Combine the estimates in (8.28) and (8.30) to obtain

$$\begin{aligned} \mathcal{E}_1^{(m)} &= 2\pi\beta \left(\beta^{-1/2} \sum_{i=1}^N (m + \overline{B}_{T_i})\Theta_i \right)^2 \le 2\pi\beta O(\varepsilon) \left(\beta^{-1/2} \sum_{i=1}^N (m + \overline{B}_{T_i})\Theta_i \right) \\ &\le O(\varepsilon)\Big(\beta^{1/3} + \beta Y_1 + Y_2 + \big(\mathcal{E}_2 + \chi - \mathcal{E}_3\big)\Big). \end{aligned} \tag{8.31}$$

Insert this estimate into (8.18) and drop the term $\mathcal{E}_3$, to find

$$Y_3^{(m)} \le O(\varepsilon)\big(\beta^{1/3} + \beta Y_1 + Y_2\big) + [1 + O(\varepsilon)]\big(\mathcal{E}_2 + \chi\big). \tag{8.32}$$

3. Estimation of m^2. Next we estimate m^2, which will be needed later. Write

$$m = \frac{1}{2\pi}\sum_{i=1}^{N}(m + \overline{B_{T_i}}\,)\,\Theta_i - \frac{1}{2\pi}\sum_{i=1}^{N}\overline{B_{T_i}}\Theta_i \tag{8.33}$$

and use $(a-b)^2 \le 2(a^2+b^2)$, to estimate

$$m^2 \le 2\left(\frac{1}{2\pi}\sum_{i=1}^{N}(m + \overline{B_{T_i}}\,)\Theta_i\right)^2 + 4\pi\left(\sum_{i=1}^{N}\overline{B_{T_i}}\,\frac{\Theta_i}{2\pi}\right)^2. \tag{8.34}$$

Up to a multiplicative constant, the first term is equal to $\mathcal{E}_1^{(m)}$, which has been estimated in Step 2. For the second term we use Cauchy-Schwarz and (8.22). Hence

$$\begin{aligned} m^2 &\le \frac{1}{\pi}\mathcal{E}_1^{(m)} + \frac{1}{\pi}\sum_{i=1}^{N}\overline{B_{T_i}}^2\Theta_i \le \frac{1}{\pi}\big(\mathcal{E}_1^{(m)} + \mathcal{E}_2 + \chi\big) \\ &\le O(\varepsilon)\big(\beta^{1/3} + \beta Y_1 + Y_2\big) + [1 + O(\varepsilon)]\big(\mathcal{E}_2 + \chi\big). \end{aligned} \tag{8.35}$$

4. Elimination of $\mathcal{E}_2$ with the help of the centreing constraint. Next we exploit the centreing constraint and the a priori estimates to get rid of $\mathcal{E}_2 = D_1^2 + D_1^{*2}$. We estimate D_1^2 only, since D_1^{*2} can be treated analogously. We have

$$D_1 = \frac{1}{\sqrt{\pi}}\left(\sum_{i=1}^{N}\overline{(m + B_{T_i})\cos T_i}\,\Theta_i - m\sum_{i=1}^{N}\overline{\cos T_i}\,\Theta_i\right). \tag{8.36}$$

Hence

$$D_1^2 \le \frac{2}{\pi}\left(\sum_{i=1}^{N}\overline{(m + B_{T_i})\cos T_i}\,\Theta_i\right)^2 + \frac{2}{\pi}\left(\sum_{i=1}^{N}\overline{\cos T_i}\,\Theta_i\right)^2 m^2. \tag{8.37}$$

In the first sum, we use the a priori estimate $\rho_i = ([\kappa-1]\beta)^{-1/2}(m + B_{T_i}) = O(\varepsilon)$. In the second sum, we use that by (5.56) and the a priori estimate $\theta_i = O(\sqrt{\varepsilon}\,)$ we have $\sum_{i=1}^{N}\overline{\cos T_i}\,\Theta_i = O(Y_1) = O(\varepsilon)$. Hence

$$D_1^2 \le O(\varepsilon)\beta^{1/2}\sum_{i=1}^{N}\overline{(m + B_{T_i})\cos T_i}\,\Theta_i + O(\varepsilon^2)m^2. \tag{8.38}$$

The term m^2 appearing in the right-hand side has been estimated in (8.35). For the first term, we exploit the centreing constraint. Define

$$Y_4^{(m)} = \sum_{i=1}^{N}(m + \overline{B_{T_i}}\,)\,\Theta_i. \tag{8.39}$$

which has been already estimated in (8.30). By Proposition 5.9, recalling Definition 5.3 and the notation in Proposition 6.6, we have $\mathcal{C}(Z^{(m)}) = (\Sigma_1, \Sigma_2)$ with

$$\Sigma_1 = \frac{1}{\pi}\frac{1}{\sqrt{(\kappa-1)\beta}}Y_5^{(m)} + O\Big(Y_1 + \frac{1}{\beta}Y_2 + \frac{1}{\beta}Y_3^{(m)} + \frac{1}{\sqrt{\beta}}|Y_4^{(m)}|\Big), \tag{8.40}$$

where

$$Y_5^{(m)} = \sum_{i=1}^{N}\overline{(m + B_{T_i})\cos T_i}\,\Theta_i. \tag{8.41}$$

By (8.40), on the event that $|\Sigma_1| \leq C\beta^{-2/3}$ we have

$$\beta^{-1/2} Y_5^{(m)} \leq O(\beta^{-2/3}) + O\Big(Y_1 + \beta^{-1} Y_2 + \beta^{-1} Y_3^{(m)} + \beta^{-1/2} |Y_4^{(m)}|\Big). \tag{8.42}$$

Multiply both sides by β and combine with (8.38), to get

$$D_1^2 \leq O(\varepsilon)\Big(\beta^{1/3} + \beta Y_1 + Y_2 + Y_3^{(m)} + \beta^{1/2} |Y_4^{(m)}|\Big) + O(\varepsilon^2)\, m^2. \tag{8.43}$$

A similar estimate holds for D_1^{*2}. Combine (8.43) with the bounds in (8.28), (8.30), (8.32) and (8.35) for $Y_3^{(m)}$, $|Y_4^{(m)}|$ and m^2, to obtain

$$\mathcal{E}_2 \leq O(\varepsilon)\big(\beta^{1/3} + \beta Y_1 + Y_2 + \chi\big) + O(\varepsilon)\,\mathcal{E}_2. \tag{8.44}$$

We subtract $O(\varepsilon)\,\mathcal{E}_2$ on both sides and multiply by $[1-O(\varepsilon)]^{-1} = 1 + O(\varepsilon)$, to conclude that we may drop the term $O(\varepsilon)\,\mathcal{E}_2$ from (8.44). Finally, we insert the estimate thus obtained into the bound (8.32) for $Y_3^{(m)}$, to find

$$Y_3^{(m)} \leq O(\varepsilon)\big(\beta^{1/3} + \beta Y_1 + Y_2\big) + [1 + O(\varepsilon)]\,\chi. \tag{8.45}$$

5. Only small m contribute. From (8.35) and (8.44) we get

$$m^2 \leq M^\varepsilon, \qquad M^\varepsilon = O(\varepsilon)\big(\beta^{1/3} + \beta Y_1 + Y_2\big) + [1 + O(\varepsilon)]\,\chi. \tag{8.46}$$

We may think of M^ε as a random variable that is typically of order $\beta^{1/3}$, so that m is typically of order $\beta^{1/6}$ (at most). However, we need to estimate the contribution of the event that M^ε is much larger than its typical order of magnitude. Fix $C' > 0$ (to be chosen later). By (8.46), we have

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{d}m\, \mathbf{1}_{\{m^2 > C'\beta^{1/3}/\varepsilon\}} \widehat{\mathbb{E}}\Big[\mathrm{e}^{O(\varepsilon)(\beta Y_1 + Y_2 + N) + \frac{1}{2} Y_3^{(m)}} \mathbf{1}_{\{Z^{(m)} \in \Upsilon^{\mathrm{UB}'}_{\beta,\varepsilon}\}}\Big] \\
&\quad \leq \widehat{\mathbb{E}}\bigg[\int_{\mathbb{R}} \mathrm{d}m\, \mathbf{1}_{\{C'\beta^{1/3}/\varepsilon < m^2 \leq M^\varepsilon\}} \mathrm{e}^{O(\varepsilon)(\beta Y_1 + Y_2 + N) + \frac{1}{2} Y_3^{(m)}} \mathbf{1}_{\{Z^{(m)} \in \Upsilon^{\mathrm{UB}'}_{\beta,\varepsilon}\}}\bigg] \\
&\quad \leq \widehat{\mathbb{E}}\bigg[\int_{\mathbb{R}} \mathrm{d}m\, \mathbf{1}_{\{C'\beta^{1/3}/\varepsilon < m^2 \leq M^\varepsilon\}} \mathrm{e}^{O(\varepsilon)(\beta^{1/3} + \beta Y_1 + Y_2 + N) + \frac{1}{2}[1 + O(\varepsilon)]\,\chi} \mathbf{1}_{\{Z^{(m)} \in \Upsilon^{\mathrm{UB}'}_{\beta,\varepsilon}\}}\bigg].
\end{aligned} \tag{8.47}$$

In the last line we first use the bound on $Y_3^{(m)}$ from (8.45) and then drop the indicator on m. The exponential in the last line is independent of m. We bound $\mathbf{1}_{\{C'\beta^{1/3}/\varepsilon < m^2 \leq M^\varepsilon\}} \leq \mathbf{1}_{\{M^\varepsilon > C'\beta^{1/3}/\varepsilon\}} \mathbf{1}_{\{m^2 \leq M^\varepsilon\}}$ and perform the integral over m, to find that we can further bound (8.47) by

$$\widehat{\mathbb{E}}\Big[\mathbf{1}_{\{M^\varepsilon > C'\beta^{1/3}/\varepsilon\}} \sqrt{M^\varepsilon}\, \mathrm{e}^{O(\varepsilon)(\beta^{1/3} + \beta Y_1 + Y_2 + N) + \frac{1}{2}[1 + O(\varepsilon)]\,\chi} \mathbf{1}_{\{Z^{(m)} \in \Upsilon^{\mathrm{UB}'}_{\beta,\varepsilon}\}}\Big]. \tag{8.48}$$

We can get rid of $\sqrt{M_\varepsilon}$ via the inequality $x \leq \mathrm{e}^{x-1}$, $x \in \mathbb{R}$, with $x = \varepsilon M_\varepsilon$, which yields $\sqrt{M_\varepsilon} \leq \frac{1}{\sqrt{\varepsilon \mathrm{e}}} \exp(\frac{1}{2}\varepsilon\, M^\varepsilon)$. Because of (8.18), the term $\frac{1}{2}\varepsilon M^\varepsilon$ can be absorbed into the exponential. We can get rid of the indicator of the event $\{M^\varepsilon > C'\beta^{1/3}/\varepsilon\}$ by estimating $\mathbf{1}_{\{M^\varepsilon > C'\beta^{1/3}/\varepsilon\}} \leq \exp(-C'\beta^{1/3} + \varepsilon M^\varepsilon)$ and again absorbing the term $\varepsilon M^\varepsilon$ into the exponential. Thus (8.48) is bounded by

$$\frac{1}{\sqrt{\varepsilon \mathrm{e}}}\, \mathrm{e}^{-C'\varepsilon\beta^{1/3}} \widehat{\mathbb{E}}\Big[\mathrm{e}^{O(\varepsilon)(\beta^{1/3} + \beta Y_1 + Y_2 + N) + \frac{1}{2}[1 + O(\varepsilon)]\,\chi}\Big]. \tag{8.49}$$

Using Hölder's inequality and Lemmas 7.2, 7.4 and 7.5, we get

$$\limsup_{\beta \to \infty} \frac{1}{\beta^{1/3}} \log \widehat{\mathbb{E}}\Big[\mathrm{e}^{O(\varepsilon)(\beta^{1/3} + \beta Y_1 + Y_2 + N) + \frac{1}{2}[1 + O(\varepsilon)]\,\chi}\Big] \leq k(\varepsilon) \tag{8.50}$$

for some $k(\varepsilon) < \infty$. The details are similar to Steps 6–7 below and therefore are omitted. Hence, given $\varepsilon > 0$ we can make (8.49) arbitrarily small by making C' sufficiently large. Altogether we obtain the following statement of exponential tightness: For every $C''>0$ there exists a $C'=C'(\varepsilon,C,C'')>0$ such that

$$\limsup_{\beta\to\infty}\frac{1}{\beta^{1/3}}\log\left(\int_{\mathbb{R}}\mathrm{d}m\,\mathbf{1}_{\{m^2>C'\beta^{1/3}/\varepsilon\}}\widehat{\mathbb{E}}\Big[\mathrm{e}^{O(\varepsilon)(\beta Y_1+Y_2+N)+\frac12 Y_3^{(m)}}\mathbf{1}_{\{Z^{(m)}\in\Upsilon^{\mathrm{UB}'}_{\beta,\varepsilon}\}}\Big]\right)\le -C''. \tag{8.51}$$

Hence we need only estimate contributions coming from $|m|\le\sqrt{C'/\varepsilon}\,\beta^{1/6}$.

6. Separation of terms with the Hölder inequality. By (8.45) and the Hölder inequality, we have for all $c>0$ and $p,q\ge 1$ with $p^{-1}+q^{-1}=1$,

$$\begin{aligned}\log\widehat{\mathbb{E}}&\left[\int_{|m|\le c\beta^{1/6}}\mathrm{d}m\,\mathrm{e}^{O(\varepsilon)(\beta Y_1+Y_2+N)+\frac12 Y_3^{(m)}}\mathbf{1}_{\{Z^{(m)}\in\Upsilon^{\mathrm{UB}'}_{\beta,\varepsilon}\}}\right]\\&\le\frac1p\sup_{|m|\le c\beta^{1/6}}\log\widehat{\mathbb{E}}\Big[\mathrm{e}^{\frac12 p[1+O(\varepsilon)]\chi}\mathbf{1}_{\{Z^{(m)}\in\Upsilon^{\mathrm{UB}'}_{\beta,\varepsilon}\}}\Big]\\&\qquad+\frac1q\log\widehat{\mathbb{E}}\Big[\mathrm{e}^{qO(\varepsilon)(\beta^{1/3}+\beta Y_1+Y_2+N)}\Big]+O(\varepsilon\beta^{1/3}).\end{aligned} \tag{8.52}$$

(We have dropped the indicator in the second term, because it will not be needed.) We will want to choose p close to 1, which makes q large and potentially dangerous for the second term in (8.52). It will turn out that a good choice is

$$q=\frac{c}{\sqrt{\varepsilon}},\qquad c\in(0,\infty). \tag{8.53}$$

for which $p=1+O(\sqrt{\varepsilon}\,)$.

7. Estimation of the second term in (8.52)**.** Note that Y_1 depends on N and Θ_i, $1\le i\le N$, alone. The tilt by $\mathrm{e}^{Y_0-\beta C_1 Y_1}$ in the definition of $\widehat{\mathbb{P}}$ affects the angular point process only, so it is still true under $\widehat{\mathbb{P}}$ that the distribution of $(B_t)_{t\in[0,2\pi]}$ is a mean-centred Brownian bridge independent from N and the Θ_i, $1\le i\le N$. Therefore, by (6.40) and Lemma 7.5, for every s such that $s\in(-1,1)$,

$$\widehat{\mathbb{E}}\big[\mathrm{e}^{\frac12 sY_2}\mid N,(\Theta_i)_{i=1}^N\big]=(1-s)^{-(N-1)/2}\quad\widehat{\mathbb{P}}\text{-a.s.} \tag{8.54}$$

Applying this identity with $s=2qO(\varepsilon)=O(\sqrt{\varepsilon}\,)$ (which falls in $(-1,1)$ for ε sufficiently small), we get

$$\widehat{\mathbb{E}}\Big[\mathrm{e}^{qO(\varepsilon)Y_2}\mid N,(\Theta_i)_{i=1}^N\Big]=e^{O(\varepsilon N)}\quad\widehat{\mathbb{P}}\text{-a.s.} \tag{8.55}$$

Multiplying both sides by $\mathrm{e}^{qO(\varepsilon)(\beta Y_1+N)}$ and taking expectations, we find

$$\frac1q\log\widehat{\mathbb{E}}\Big[\mathrm{e}^{qO(\varepsilon)(\beta Y_1+Y_2+N)}\Big]\le\frac1q\log\widehat{\mathbb{E}}\Big[\mathrm{e}^{qO(\varepsilon)(\beta Y_1+N)}\Big]. \tag{8.56}$$

It now follows from Lemma 7.2 with $\delta=q\varepsilon=c\sqrt{\varepsilon}$ that

$$\limsup_{\beta\to\infty}\frac{1}{\beta^{1/3}}\frac1q\log\widehat{\mathbb{E}}\Big[\mathrm{e}^{qO(\varepsilon)(\beta Y_1+Y_2+N)}\Big]\le O(\varepsilon). \tag{8.57}$$

Hence the second term in (8.52) is negligible.

8. Conclusion. Combine (8.51)–(8.52) and (8.57) to get

$$\begin{aligned}\limsup_{\beta\to\infty}&\frac{1}{\beta^{1/3}}\log\left(\int_{\mathbb{R}}\mathrm{d}m\;\widehat{\mathbb{E}}\Big[\mathrm{e}^{O(\varepsilon)(\beta Y_1+Y_2+N)+\frac12 Y_3^{(m)}}\mathbf{1}_{\{Z^{(m)}\in\Upsilon^{\mathrm{UB}'}_{\beta,\varepsilon}\}}\Big]\right)\\&\le O(\varepsilon)+\limsup_{\beta\to\infty}\frac{1}{\beta^{1/3}}\sup_{|m|\le c\beta^{1/6}}\log\widehat{\mathbb{E}}\Big[\mathrm{e}^{\frac12[1+O(\varepsilon)]\,\chi}\mathbf{1}_{\{Z^{(m)}\in\mathcal{O}\}}\Big]\end{aligned} \tag{8.58}$$

for suitable $c=c(\varepsilon,C)>0$. Together with the representation (6.41) for the key integral and Proposition 7.1, this completes the proof of Proposition 8.1.

8.2 Lower bound

The proof of the lower bound in Theorem 2.5 builds on the following key proposition.

Proposition 8.2. *For all $C \in (0,\infty)$ sufficiently large and all $\varepsilon > 0$ sufficiently small,*

$$\limsup_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log \mathcal{I}^{\mathrm{LB}}(\kappa,\beta;C,\varepsilon) \geq 2\pi G_\kappa \tau_* + \liminf_{\beta\to\infty} \inf_{|m|\leq\beta^{-1/6}} \frac{1}{\beta^{1/3}} \log \widehat{\mathbb{P}}\big(Z^{(m)} \in \mathcal{O}\big) - O(\varepsilon). \tag{8.59}$$

Proof of the lower bound in Theorem 2.5. The lower bound in Theorem 2.5 follows from Corollary 6.4, Proposition 8.2 and conditions (C1) and (C2) in Section 2.2. □

Proof of the lower bound in Theorem 2.6. For the proof of the lower bound in Theorem 2.6, we work directly with the surface integrals and skip the auxiliary random variables. By Corollary 6.4, we have

$$\begin{aligned} &\mathrm{e}^{\beta I^*(\pi R_c^2)} \mu_\beta\Big(\mathcal{V}_{C\beta^{-2/3}}\Big) \\ &\quad\geq \mathrm{e}^{\beta I^*(\pi R_c^2)} \mu_\beta\Big(\mathcal{V}_{C\beta^{-2/3}} \cap \mathcal{D}_\varepsilon(0)\Big) \\ &\quad\geq \Big[1 - O(\mathrm{e}^{-c\kappa\beta})\Big] (\kappa\beta)^n \int_{[0,2\pi]^n} \mathrm{d}t\, \mathbf{1}_{\{0\leq t_1<\cdots<t_n<2\pi\}} \int_{\mathbb{R}^n} \mathrm{d}r\, \mathrm{e}^{-\beta\Delta(z)}\, \mathbf{1}_{\mathcal{V}'_{C\beta^{-2/3}} \cap \mathcal{D}'_\varepsilon(0)}(z), \end{aligned} \tag{8.60}$$

where $n \in \mathbb{N}$ is arbitrary, $z = (z_1,\ldots,z_n)$ and $z_i = (r_i \cos t_i, r_i \sin t_i)$. We pick $n = \lfloor k\beta^{1/3} \rfloor$ with $k > 0$ some constant, write $r_i = R_c - 2 + \rho_i$, and restrict the integral to the domain

$$M = \Big\{(t,\rho)\colon\ , \max_{1\leq i\leq n} \big|t_i - \tfrac{2\pi}{n}(i - \tfrac{1}{2})\big| \leq \varepsilon_1 \frac{2\pi}{n},\ \max_{1\leq i\leq n} |\rho_i| \leq \varepsilon_2 \beta^{-2/3}\Big\} \tag{8.61}$$

with $\varepsilon_1 \in (0, \frac{1}{3})$ and $\varepsilon_2 > 0$.

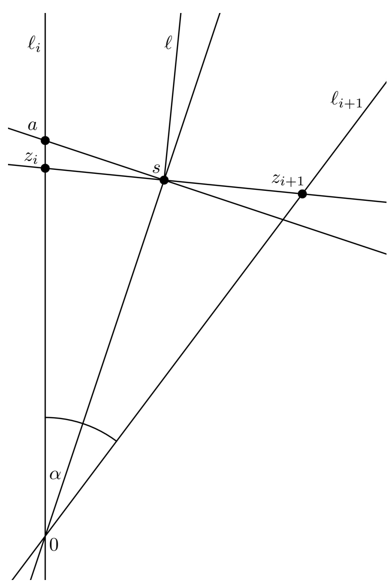

Figure 10: Illustration to the proof of Lemma 8.3: $a = (0, R_c - 2)$, and s is the barycentre of the points z_i and z_{i+1}.

Lemma 8.3. *Let $z_i \in M$, $i = 1,\ldots,n$, and let $\varepsilon_1 \leq 1/3$ and $\varepsilon_2 \leq \frac{R_c-2}{2k^2}$. Then $z \in \mathcal{O} \cap \mathcal{D}'_\varepsilon(0)$ for sufficiently large β.*

Proof. First, we prove that $z \in \mathcal{O}$. Employing Proposition 5.7, we need to show that every triplet (z_{i-1}, z_i, z_{i+1}) is extremal. Actually, we will show that the intersection of the halfline ℓ_i starting at the origin and passing through z_i is intersecting the circle $\partial B_2(z_i)$ in a point $p \notin B_2(z_{i-1}) \cup B_2(z_{i+1})$. Clearly, it suffices to show that $p \notin B_2(z_{i+1})$.

Consider the most extremal case: ρ_i attains the minimum allowed value $\rho_i = -\varepsilon_2\beta^{-2/3}$ and ρ_{i+1} attain the maximum allowed value $\rho_{i+1} = \varepsilon_2\beta^{-2/3}$. Also, take the minimal angle α between the half-lines ℓ_i and ℓ_{i+1}: $\alpha = \frac{2\pi}{n}(1-\frac{2}{3})$. Without loss of generality, we may assume that $z_i = (0, R_c - 2 - \varepsilon_2\beta^{-2/3})$ and $z_{i+1} = (r\sin\alpha, r\cos\alpha)$, where $r = R_c - 2 + \varepsilon_2\beta^{-2/3}$. Let s be the barycentre $s = \frac{1}{2}(z_i + z_{i+1})$, and let ℓ be the half-line beginning at s, orthogonal to the segment (z_i, z_{i+1}), and containing the points (x,y) with $y > R_c$ (see Fig. 10). The point $v_i \in \partial B_2(z_i) \cap \partial B_2(z_i)$ belongs to ℓ. If the half-line ℓ does not intersect the positive y-axis $\ell \cap \{(0,y)\colon y \ge 0\} = \emptyset$, then $p = (0, R_c - \varepsilon_2\beta^{-2/3}) \notin B_2(z_{i+1})$.

To show that the half-line ℓ does not intersect the positive y-axis, it suffices to show that $R_c - 2 - \varepsilon_2\beta^{-2/3} \ge s_2$, where $s_2 = \frac{1}{2}(R_c - 2 - \varepsilon_2\beta^{-2/3} + (R_c - 2 + \varepsilon_2\beta^{-2/3})\cos\alpha)$ is the second coordinate of the barycentre s. This leads to the condition $(R_c - 2)(1-\cos\alpha) \ge \varepsilon_2\beta^{-2/3}(1+\cos\alpha)$. Given that

$$\frac{1-\cos\alpha}{1+\cos\alpha} = \frac{\sin^2\alpha}{(1+\cos\alpha)^2} \ge \frac{\sin^2\alpha}{4} \ge \frac{\alpha^2}{8} \ge \frac{(2\pi/3)^2}{8}(k\beta^{1/3})^{-2} \ge \frac{1}{2k^2}\beta^{-2/3}, \tag{8.62}$$

we get the sufficient condition $\varepsilon_2 \le \frac{R_c-2}{2k^2}$, as claimed.

For the proof that $z \in \mathcal{D}'_\varepsilon(0)$, we refer to the beginning of the proof of Lemma 8.4, where we show that if $\alpha \le \frac{2\sqrt{2}}{R_c(R_c-2)}\sqrt{\varepsilon}$, then the intersecting point v_i on the boundary belongs to $A_{R_c,\varepsilon}$. The above bound on α is clearly satisfied once β is sufficiently large. □

Returning to the proof of the lower bound in Theorem 2.6, we check that the volume constraint is satisfied as well (recal the orders of magnitude of the relevant quantities from Definition 5.3). Note that for $(t,\rho) \in M$ the angular increments are bounded as

$$\frac{1}{3}\frac{2\pi}{n} \le \theta_i \le \frac{5}{3}\frac{2\pi}{n} \tag{8.63}$$

and $\frac{2\pi}{n} = [1+o(1)]\,2\pi k^{-1}\beta^{1/3}$, as $\beta \to \infty$. On M,

$$\begin{aligned}
y_1(z) &= \sum_{i=1}^n \theta_i^3 \le 2\pi \max_i \theta_i^2 \le 2\pi\,\tfrac{25}{9}\left(\frac{2\pi}{n}\right)^2 = \big[1+o(1)\big]\,\tfrac{25}{9}\,8\pi^3\,k^{-2}\,\beta^{-2/3},\\
y_2(z) &= \sum_{i=1}^n \frac{(\rho_{i+1}-\rho_i)^2}{\theta_i} \le 2\pi\,\frac{(2\varepsilon_2\beta^{-2/3})^2}{[2\pi/(3n)]^2} \le \big[1+o(1)\big]\,\tfrac{18}{\pi}\varepsilon_2^2 k^{-2}\beta^{-2/3},\\
y_3(z) &= \sum_{i=1}^n \overline{\rho}_i^2\theta_i \le \big[1+o(1)\big]2\pi\varepsilon_2^2\,\beta^{-4/3} = o(\beta^{-2/3}),\\
\big|y_4(z)\big| &\le \sum_{i=1}^n |\overline{\rho}_i|\theta_i \le 2\pi\varepsilon_2\beta^{-2/3}.
\end{aligned} \tag{8.64}$$

Choosing k large enough and ε_2 small enough, we see that each of the y_i's is of order at most $C'\,\beta^{-2/3}$, where C' can be made arbitrarily small compared to the given constant $C > 0$. By Proposition 5.8, the volume constraint is therefore satisfied for sufficiently large β. Thus, we may drop the indicators from the last line of (8.60). Proposition 5.8 and the bounds in (8.63) also yield the estimate

$$\Delta(z) \le C'\beta^{-2/3} \tag{8.65}$$

for some constant $C' > 0$ depending on k and ε_2. We deduce that

$$\begin{aligned}
\mathrm{e}^{\beta I^*(\pi R_c^2)}\mu_\beta\Big(\mathcal{V}_{C\beta^{-2/3}}\Big) &\ge (\kappa\beta)^n \mathrm{e}^{-C'\beta^{1/3}}\big(2\varepsilon_1\tfrac{2\pi}{n}\big)^n\big(\varepsilon_2\beta^{-2/3}\big)^n\\
&= \big[1+o(1)\big]\,\mathrm{e}^{-C'\beta^{1/3}}\big(\kappa\,4\pi k^{-1}\varepsilon_2\big)^n
\end{aligned} \tag{8.66}$$

and

$$\liminf_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log\Big(\mathrm{e}^{\beta I^*(\pi R_c^2)} \mu_\beta\Big(\mathcal{V}_{C\beta^{-2/3}}\Big)\Big) \geq -C' + k\log\big(\kappa\, 4\pi k^{-1}\varepsilon_2\big). \tag{8.67}$$

The right-hand side can be written as $2\pi G_\kappa(\tau_* - c)$ for some $c \geq 0$, since we have already proven the upper bound in Theorem 2.6, and the upper bound must be larger than the lower bound. This completes the proof of Theorem 2.6. $\square$

The proof of Proposition 8.2 comes in 4 steps. We again start from (6.41). For the lower bound, we can simply drop the non-negative term $Y_3^{(m)}$ and restrict the integral over m to $|m| \leq \beta^{-1/6}$.

1. Separation of terms with the reverse Hölder inequality. We separate the exponential from the indicator $\Upsilon^{\mathrm{LB}'}_{\beta,\varepsilon}$ (defined in (6.43)) with the help of the reverse Hölder inequality with $p \in (1,\infty)$,

$$\begin{aligned} &\log \widehat{\mathbb{E}}\Big[\int_{|m|\leq\beta^{-1/6}} \mathrm{d}m\, \mathrm{e}^{O(\varepsilon)(\beta Y_1 + Y_2 + N)} \mathbf{1}_{\{Z^{(m)}\in\Upsilon^{\mathrm{LB}'}_{\beta,\varepsilon}\}}\Big] \\ &\geq p \inf_{|m|\leq\beta^{-1/6}} \log \widehat{\mathbb{P}}\big(Z^{(m)} \in \Upsilon^{\mathrm{LB}'}_{\beta,\varepsilon}\big) - (p-1) \sup_{|m|\leq\beta^{-1/6}} \log \widehat{\mathbb{E}}\Big[\mathrm{e}^{-\frac{1}{p-1}O(\varepsilon)(\beta Y_1+Y_2+N)}\Big] + \log\beta^{-1/6}. \end{aligned} \tag{8.68}$$

We choose

$$p = 1 + c\sqrt{\varepsilon}, \qquad c \in (0,\infty). \tag{8.69}$$

2. Estimation of the second term in (8.68)**.** Proceeding as in the proof of (8.57), we can again use (8.54) with $s = -2(p-1)^{-1}O(\varepsilon) = -O(\sqrt{\varepsilon}\,)$ to estimate, as in (8.55),

$$-(p-1)\log\widehat{\mathbb{E}}\Big[\mathrm{e}^{-\frac{1}{p-1}O(\varepsilon)\,Y_2} \mid N, (\Theta_i)_{i=1}^N\Big] = (p-1)\tfrac{(N-1)}{2}\log(1-s) = O(\sqrt{\varepsilon}\,N). \tag{8.70}$$

Taking expectations, we obtain

$$-(p-1)\log\widehat{\mathbb{E}}\Big[\mathrm{e}^{-\frac{1}{p-1}O(\varepsilon)\,(\beta Y_1+Y_2+N)}\Big] = -(p-1)\log\widehat{\mathbb{E}}\Big[\mathrm{e}^{-\frac{1}{p-1}O(\varepsilon)\,(\beta Y_1+N)}\Big]. \tag{8.71}$$

Applying Lemma 7.2 with $\delta = \frac{1}{p-1}O(\varepsilon) = O(\sqrt{\varepsilon}\,)$, we conclude that

$$\liminf_{\beta\to\infty} -\frac{1}{\beta^{1/3}}(p-1)\log\widehat{\mathbb{E}}\Big[\mathrm{e}^{-\frac{1}{p-1}O(\varepsilon)\,(\beta Y_1+Y_2+N)}\Big] \geq -O(\varepsilon). \tag{8.72}$$

3. Estimate of the first term in (8.68)**.** Estimate

$$\begin{aligned} &\widehat{\mathbb{P}}\big(Z^{(m)} \in \mathcal{O}\cap\mathcal{V}'_{C\beta^{-2/3}}\cap\mathcal{D}'_\varepsilon(0)\big) = \widehat{\mathbb{P}}\big(Z^{(m)}\in\mathcal{O}\big) \\ &\quad - \Big[\widehat{\mathbb{P}}\big(Z^{(m)}\in\mathcal{O},\, Z^{(m)}\notin\mathcal{D}'_\varepsilon(0)\big) + \widehat{\mathbb{P}}\big(Z^{(m)}\in\mathcal{O}\cap\mathcal{D}'_\varepsilon(0),\, Z^{(m)}\notin\mathcal{V}'_{C\beta^{-2/3}}\big)\Big]. \end{aligned} \tag{8.73}$$

We want to show that the last two probabilities are negligible. It suffices to show the following.

Lemma 8.4. *For some $\varepsilon_0 > 0$ small enough and uniformly on $|m| \leq \beta^{-1/6}$:*

(a) $\lim_{\beta\to\infty} \frac{1}{\beta^{1/3}}\log\widehat{\mathbb{P}}\big(Z^{(m)}\notin\mathcal{D}'_\varepsilon(0)\big) = -\infty$ *for every* $0<\varepsilon\leq\varepsilon_0$.

(b) $\lim_{C\to\infty}\sup_{0<\varepsilon\leq\varepsilon_0}\limsup_{\beta\to\infty}\frac{1}{\beta^{1/3}}\log\widehat{\mathbb{P}}\big(Z^{(m)}\in\mathcal{O}\cap\mathcal{D}'_\varepsilon(0),\, Z^{(m)}\notin\mathcal{V}'_{C\beta^{-2/3}}\big) = -\infty$.

Proof. (a) First, let us show that

$$\begin{aligned} &\Big\{Z^{(m)}\in\mathcal{O},\, \forall\, 1\leq i\leq N\colon\ \Theta_i \leq \frac{2\sqrt{2}}{R_c(R_c-2)}\sqrt{\epsilon},\, \forall\, 1\leq i\leq N\colon\ |m+B_{T_i}| \leq \tfrac12\varepsilon\sqrt{(\kappa-1)\beta}\Big\} \\ &\qquad\qquad \subset \big\{Z^{(m)}\in\mathcal{O}\cap\mathcal{D}'_\varepsilon(0)\big\}. \end{aligned} \tag{8.74}$$

In polar coordinates, $Z_i^{(m)}=(r_i^{(m)}\cos T_i, r_i^{(m)}\sin T_i)$ with $|r_i^{(m)}-(R_c-2)|\leq\frac{1}{2}\varepsilon$. To show that $Z^{(m)}\in\mathcal{O}\cap\mathcal{D}'_\varepsilon(0)$, it clearly suffices to show that, for any i, the boundary point $v_i\in\partial B_2(Z_i^{(m)})\cap\partial B_2(Z_{i+1}^{(m)})$ belongs to $A_{R_c,\varepsilon}$. The most extremal case occurs when $|r_i^{(m)}|=|r_{i+1}^{(m)}|=r=R_c-2-\frac{1}{2}\varepsilon$ and $\Theta_i=\frac{2\sqrt{2}}{R_c(R_c-2)}\sqrt{\varepsilon}$. Assuming, without loss of generality, that $Z_i^{(m)}=(0,r)$ and $Z_{i+1}^{(m)}=(r\sin\Theta_i, r\cos\Theta_i)$, we find v_i as the intersection of the line $\{(x,y): x=y\tan(\Theta_i/2)\}$ (the axis of symmetry between points $Z_i^{(m)}$ and $Z_{i+1}^{(m)}$) and the circle $\partial B_2((0,r))$. we get

$$|v_i|^2=(r+2)^2-(2r+2r^2+r^3/2)\tfrac{\Theta_i^2}{4}+O(\varepsilon^{3/2})\geq(R_c-\varepsilon)^2. \tag{8.75}$$

For the last inequality, notice that

$$(2r+2r^2+r^3/2)\tfrac{\Theta_i^2}{4}=\tfrac{r}{2}(r+2)^2\tfrac{\Theta_i^2}{4}\leq(r+2)\varepsilon \tag{8.76}$$

using the assumption $\Theta_i\leq\frac{2\sqrt{2}}{R_c(R_c-2)}\sqrt{\varepsilon}$ and the fact that $r+2<R_c$.

With the help of the proven inclusion, we may estimate

$$\begin{aligned}\widehat{\mathbb{P}}\Big(Z^{(m)}\in\mathcal{O},\, Z^{(m)}\notin\mathcal{D}'_\varepsilon(0)\Big)\leq\widehat{\mathbb{P}}\Big(\exists\, 1\leq i\leq N\colon\ \Theta_i>\frac{2\sqrt{2}}{R_c(R_c-2)}\sqrt{\varepsilon}\Big)&\\ +\widehat{\mathbb{P}}\Big(\exists\, 1\leq i\leq N\colon\ |m+B_{T_i}|>\tfrac{1}{2}\varepsilon\sqrt{(\kappa-1)\beta}\Big)&.\end{aligned} \tag{8.77}$$

Abbreviate $k=2\sqrt{2}/R_c(R_c-2)$. Then the first term in the right-hand side of (8.77) can be estimated as

$$\widehat{\mathbb{P}}\Big(\exists\, 1\leq i\leq N\colon\ \Theta_i>k\sqrt{\varepsilon}\Big)\leq\frac{1}{\mathbb{E}[\exp(\widehat{Y}_0-\widehat{Y}_1)]}\,\mathbb{E}\Bigg[\sum_{i=1}^N\mathbb{E}\Big[\mathrm{e}^{\widehat{Y}_0-\widehat{Y}_1}1_{\{\Theta_i>k\sqrt{\varepsilon}\}}\,\Big|\,N\Big]\Bigg]. \tag{8.78}$$

In the conditional expectation we estimate

$$\widehat{Y}_1=\beta C_1\sum_{j=1}^n\Theta_j^3\geq\tfrac{1}{2}\Theta_i^3+\tfrac{1}{2}\widehat{Y}_1\geq\tfrac{1}{2}k^3\varepsilon^{3/2}+\tfrac{1}{2}\widehat{Y}_1\quad\mathbb{P}\text{-a.s.} \tag{8.79}$$

With the help of the inequality $N\leq\frac{1}{\delta\mathrm{e}}\mathrm{e}^{N\delta}$ we deduce that, for every $\delta>0$,

$$\begin{aligned}\widehat{\mathbb{P}}\Big(\exists\, 1\leq i\leq N\colon\ \Theta_i>k\sqrt{\varepsilon}\Big)&\leq\mathrm{e}^{-k^3\varepsilon^{3/2}\beta}\,\frac{\mathbb{E}[N\,\exp(\widehat{Y}_0-\frac{1}{2}\widehat{Y}_1)]}{\mathbb{E}[\exp(\widehat{Y}_0-\widehat{Y}_1)]}\\ &\leq\frac{1}{\delta\mathrm{e}}\mathrm{e}^{-k^3\varepsilon^{3/2}\beta}\,\frac{\mathbb{E}[\exp(\widehat{Y}_0-\frac{1}{2}\widehat{Y}_1+\delta N)]}{\mathbb{E}[\exp(\widehat{Y}_0-\widehat{Y}_1)]}.\end{aligned} \tag{8.80}$$

We already know from Proposition 7.1 that the denominator equals $\exp(-[1+o(1)]\,c\beta^{1/3})$ for some constant $c>0$ as $\beta\to\infty$. Arguments entirely analogous to those in the proof of Proposition 7.1 and Lemma 7.2 show that the same holds for the numerator. It follows that

$$\widehat{\mathbb{P}}\Big(\exists\, 1\leq i\leq N\colon\ \Theta_i>k\sqrt{\varepsilon}\Big)=O\big(\mathrm{e}^{-c'\varepsilon^{3/2}\beta}\big) \tag{8.81}$$

for some constant $c'>0$ as $\beta\to\infty$, and so we get the claim with a margin.

As to the second term in the right-hand side of (8.77), since the tilting only affects the angular process and not the radial process, we have

$$\begin{aligned}\widehat{\mathbb{P}}\Big(\exists\, 1\leq i\leq N\colon\ |m+B_{T_i}|>\tfrac{1}{2}\varepsilon\sqrt{(\kappa-1)\beta}\Big)&=\sum_{n\in\mathbb{N}}\widehat{\mathbb{E}}\Big(1_{\{\exists\,1\leq i\leq n\colon\ |m+B_{T_i}|>\frac{1}{2}\varepsilon\sqrt{(\kappa-1)\beta},\,N=n\}}\Big)\\ \leq\sum_{n\in\mathbb{N}}\sum_{i=1}^n\widehat{\mathbb{E}}\Big(1_{\{|m+B_{T_i}|>\frac{1}{2}\varepsilon\sqrt{(\kappa-1)\beta},\,N=n\}}\Big)&=\sum_{n\in\mathbb{N}}\sum_{i=1}^n\widehat{\mathbb{E}}\Big(1_{\{|m+B_0|>\frac{1}{2}\varepsilon\sqrt{(\kappa-1)\beta},\,N=n\}}\Big)\\ =\widehat{\mathbb{E}}[N]\,\widehat{\mathbb{P}}\big(|m+B_0|>\tfrac{1}{2}\varepsilon\sqrt{(\kappa-1)\beta}\,\big)&=\widehat{\mathbb{E}}[N]\,\mathbb{P}\big(|m+B_0|>\tfrac{1}{2}\varepsilon\sqrt{(\kappa-1)\beta}\,\big),\end{aligned} \tag{8.82}$$

where we use that the law of $(B_t)_{t\in[0,2\pi]}$ is invariant under shifts. Moreover, recalling (2.17)–(2.18), we have

$$\begin{aligned}\mathbb{P}\big(|m+B_0|>\tfrac12\varepsilon\sqrt{(\kappa-1)\beta}\,\big)&=\mathbb{P}\left(\left|\tfrac{1}{2\pi}\int_0^{2\pi}\mathrm{d}t\,\widetilde{W}_t\right|>\tfrac12\varepsilon\sqrt{(\kappa-1)\beta}-|m|\right)\\&\le\mathbb{P}\left(\sup_{0\le t\le 2\pi}|W_t|>\tfrac23\left[\tfrac12\varepsilon\sqrt{(\kappa-1)\beta}-|m|\right]\right),\end{aligned}\tag{8.83}$$

where we use that $\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}t\,\widetilde{W}_t=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}s\,W_s-\frac12 W_{2\pi}$ and

$$\left\{\sup_{0\le t\le 2\pi}|W_t|\le a\right\}\subset\left\{\left|\tfrac{1}{2\pi}\int_0^{2\pi}\mathrm{d}s\,W_s-\tfrac12 W_{2\pi}\right|\le\tfrac32 a\right\},\qquad a>0.\tag{8.84}$$

But (see [11, Lemma 5.2.1])

$$\mathbb{P}\left(\sup_{0\le t\le 2\pi}|W_t|>\tfrac23\left[\tfrac12\varepsilon\sqrt{(\kappa-1)\beta}-|m|\right]\right)\le 4\exp\left(-\tfrac12\tfrac{1}{2\pi}\left[\tfrac23\left[\tfrac12\varepsilon\sqrt{(\kappa-1)\beta}-|m|\right]\right]^2\right),\tag{8.85}$$

and so we get the claim with a margin.

(b) On the event $\{Z^{(m)}\in\mathcal{O}\cap\mathcal{D}'_\varepsilon(0)\}$ we can use the expansion of the volume in Proposition 5.8 in the form given in (8.23), with $Y_4^{(m)}$ given in (8.39). We have

$$|S(Z^{(m)})|-\pi R_c^2=-C_1^\varepsilon Y_1+\frac{1+O(\varepsilon)}{2\beta}Y_2+\frac12\frac{1}{(\kappa-1)\beta}Y_3^{(m)}+\frac{R_c}{\sqrt{(\kappa-1)\beta}}Y_4^{(m)}.\tag{8.86}$$

Recall that $Y_1,Y_2,Y_3^{(m)}$ are non-negative, while Y_4 is not necessarily so. It follows that

$$\begin{aligned}\widehat{\mathbb{P}}\big(Z^{(m)}\in\mathcal{O}\cap\mathcal{D}'_\varepsilon(0),\,Z^{(m)}\notin\mathcal{V}'_{C\beta^{-2/3}}\big)&\le\widehat{\mathbb{P}}\Big(C_1^\varepsilon Y_1>\tfrac14 C\beta^{-2/3}\Big)+\widehat{\mathbb{P}}\Big(\tfrac{1}{2\beta}[1+O(\varepsilon)]\,Y_2>\tfrac14 C\beta^{-2/3}\Big)\\&+\widehat{\mathbb{P}}\Big(\tfrac12\frac{1}{(\kappa-1)\beta}\,Y_3^{(m)}>\tfrac14 C\beta^{-2/3}\Big)+\widehat{\mathbb{P}}\Big(\frac{R_c}{\sqrt{(\kappa-1)\beta}}\,|Y_4^{(m)}|>\tfrac14 C\beta^{-2/3}\Big).\end{aligned}\tag{8.87}$$

The four probabilities on the right-hand side of (8.87) are estimated with the help of large deviations, Markov's inequality and the results from Section 7. The first probability is bounded by

$$\exp\big(-\tfrac14 sC\beta^{1/3}\big)\,\widehat{\mathbb{E}}\Big[\mathrm{e}^{sC_1^\varepsilon\beta Y_1}\Big],\quad s>0.\tag{8.88}$$

Using Lemma 7.2, we see that for $s\downarrow 0$,

$$\limsup_{\beta\to\infty}\frac{1}{\beta^{1/3}}\log\widehat{\mathbb{E}}\Big[\mathrm{e}^{sC_1^\varepsilon\beta Y_1}\Big]<O(s)\tag{8.89}$$

and therefore, for some $\varepsilon_1>0$,

$$\lim_{C\to\infty}\sup_{0<\varepsilon\le\varepsilon_1}\limsup_{\beta\to\infty}\frac{1}{\beta^{1/3}}\log\widehat{\mathbb{P}}\Big(C_1^\varepsilon Y_1>\tfrac14 C\beta^{-2/3}\Big)=-\infty.\tag{8.90}$$

The other three probabilities in (8.87) are treated in a similar way, and so it suffices to show that, for $s\in\mathbb{R}$ with $|s|$ small enough,

$$\limsup_{\beta\to\infty}\frac{1}{\beta^{1/3}}\log\widehat{\mathbb{E}}\big[\mathrm{e}^{sY_2}\big]<\infty,\ \limsup_{\beta\to\infty}\frac{1}{\beta^{1/3}}\log\widehat{\mathbb{E}}\big[\mathrm{e}^{sY_3^{(m)}}\big]<\infty,\ \limsup_{\beta\to\infty}\frac{1}{\beta^{1/3}}\log\widehat{\mathbb{E}}\big[\mathrm{e}^{s\beta^{1/2}Y_4^{(m)}}\big]<\infty.\tag{8.91}$$

First term. Write Use (8.54) to compute

$$\widehat{\mathbb{E}}\big(\mathrm{e}^{sY_2} \mid N, (\Theta_i)_{i=1}^N\big) = \big(1-2s\big)^{-(N-1)/2} = \exp\left(\tfrac{N-1}{2}\log(1-2s)^{-1}\right) \le \mathrm{e}^{N[s+O(s^2)]} \quad \widehat{\mathbb{P}}\text{-a.s.} \tag{8.92}$$

Taking the expectation, we have

$$\widehat{\mathbb{E}}\big(\mathrm{e}^{sY_2}\big) \le \widehat{\mathbb{E}}\big(\mathrm{e}^{[s+O(s^2)]N}\big) \tag{8.93}$$

and therefore, using Lemma 7.2, we see that for $s \downarrow 0$,

$$\limsup_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log \widehat{\mathbb{E}}\big(\mathrm{e}^{sY_2}\big) \le O(s). \tag{8.94}$$

Second term. Note that $Y_3^{(m)} = \sum_{i=1}^N (m+\overline{B_{T_i}})^2\Theta_i \le \sum_{i=1}^N 2(m^2+\overline{B_{T_i}}^2)\Theta_i = 4\pi m^2 + 2Y_3^{(0)}$. The term $4\pi m^2 = O(\beta^{-1/3})$ is harmless. Write

$$Y_3^{(0)} = E_1 + \int_0^{2\pi} \mathrm{d}t\, B_t^2, \qquad E_1 = \sum_{i=1}^N \overline{B_{T_i}}^2\Theta_i - \int_0^{2\pi} \mathrm{d}t\, B_t^2. \tag{8.95}$$

Estimate

$$\widehat{\mathbb{E}}\big(\mathrm{e}^{sY_3^{(0)}}\big) \le \widehat{\mathbb{E}}\big(\mathrm{e}^{2sE_1}\big)^{\frac12}\, \widehat{\mathbb{E}}\big(\mathrm{e}^{2s\int_0^{2\pi}\mathrm{d}t\,B_t^2}\big)^{\frac12}. \tag{8.96}$$

By Lemma 7.8,

$$\widehat{\mathbb{E}}\big(\mathrm{e}^{2sE_1}\big) \le \widehat{\mathbb{E}}\Big(\mathrm{e}^{|s|O\left(\sqrt{Y_1}\log(1/Y_1)\right)}\Big). \tag{8.97}$$

Since $Y_1 \ge N(2\pi/N)^3$, we have $\log(1/Y_1) = O(\log N) = O(N)$. Since $Y_1 = O(\varepsilon)$, this gives

$$\widehat{\mathbb{E}}\big(\mathrm{e}^{2sE_1}\big) \le \widehat{\mathbb{E}}\big(\mathrm{e}^{|s|O(\sqrt{\varepsilon}\log N)}\big) = e^{|s|O(\sqrt{\varepsilon})\beta^{1/3}}, \tag{8.98}$$

where we use Lemma 7.2. By Lemma 7.4,

$$\widehat{\mathbb{E}}\big[\mathrm{e}^{2s\int_0^{2\pi}\mathrm{d}t\,B_t^2}\big] = \prod_{k\in\mathbb{N}}\Big(1-\frac{4s}{k^2}\Big)^{-1} \tag{8.99}$$

which is finite for $s < \frac14$.

Third term. Note that $Y_4^{(m)} = \sum_{i=1}^N (m+\overline{B_{T_i}})\Theta_i = 2\pi m + Y_4^{(0)}$. The term $2\pi m = O(\beta^{-1/6})$ is again harmless. We have

$$\widehat{\mathbb{E}}\big(\mathrm{e}^{s\beta^{1/2}Y_4^{(0)}}\big) = \widehat{\mathbb{E}}\left(\exp\left[s\beta^{1/2}\sum_{i=1}^N \overline{B_{T_i}}\,\Theta_i\right]\right). \tag{8.100}$$

Use Lemma 7.7 to bound this from above by

$$\widehat{\mathbb{E}}\left(\exp\left[\tfrac{1}{32\pi}s^2\beta Y_1\right]\right). \tag{8.101}$$

Now use (8.94) to get

$$\limsup_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log \widehat{\mathbb{E}}\big(\mathrm{e}^{s\beta^{1/2}Y_4^{(0)}}\big) \le O(s). \tag{8.102}$$

This completes the proof of (8.91). □

4. Conclusion. Combining (8.68), (8.72), (8.73) and Lemma 8.4, and choosing C large enough, we get

$$\begin{aligned}
&\liminf_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log \widehat{\mathbb{E}}\Big[\int_{|m|\le\beta^{-1/6}} \mathrm{d}m\, \mathrm{e}^{O(\varepsilon)(\beta Y_1+Y_2+N)}\mathbf{1}_{\{Z^{(m)}\in\Upsilon^{\mathrm{LB}'}_{\beta,\varepsilon}\}}\Big] \\
&\qquad \ge \liminf_{\beta\to\infty} \inf_{|m|\le\beta^{-1/6}} \frac{1}{\beta^{1/3}} \log \widehat{\mathbb{P}}\big(Z^{(m)} \in \mathcal{O}\big) - O(\varepsilon).
\end{aligned} \tag{8.103}$$

Proposition 8.2 follows with (6.41) and Proposition 7.1.

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